17 August 2023 21:10

Double Integrals

Suppose we have $u = f(x, y)$.

$I = \int\int f(x,y)\, dy\, dx$ [perform integration w.r.t x & y].

Order of integration $I = \int \left\{ \int f(x,y)\, dy \right\} dx$

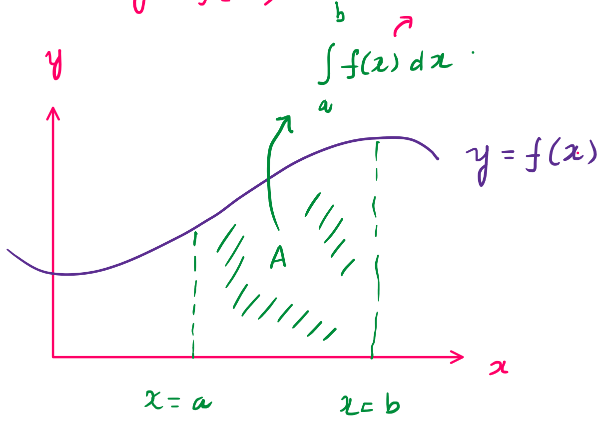
[Integrate first w.r.t y & then integrate the resultant w.r.t x].

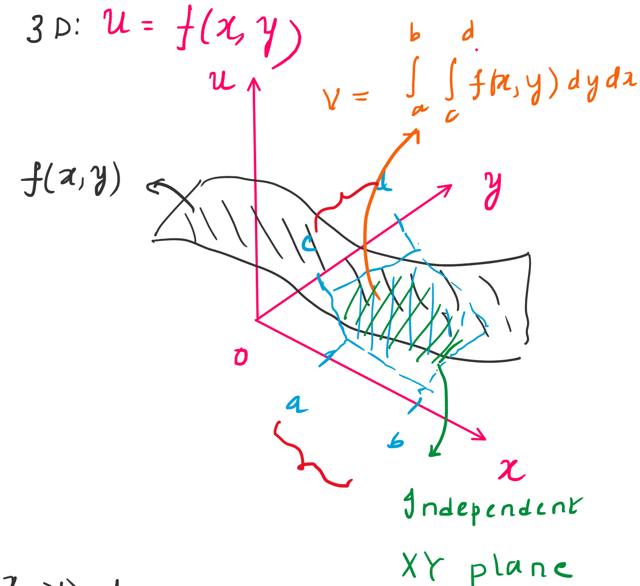
Definite integral: $I = \int_a^b \int_c^d f(x,y)\, dy\, dx$, $a \le x \le b$, $c \le y \le d$

Graphical Interpretation:

2D: $y = f(x)$

3D: $u = f(x, y)$





Note: $\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_a^b dx \int_c^d f(x,y)\, dy$

Q. $\int_0^{\pi/2} \int_0^{\pi} \sin(x+y)\, dy\, dx$, $x \in [0, \pi/2]$, $y \in [0, \pi]$

↳ Independent Range.

$\int_0^{\pi/2} dx \left\{ \int_0^{\pi} \sin(x+y)\, dy \right\}$ [Integrate w.r.t y keeping 'x' constant]

$\int_0^{\pi/2} dx \left[-\cos(x+y) \right]_0^{\pi}$

$$\int_0 dx \left[-\cos(x+y)\right]_0^{\pi}$$

$$\int_0^{\pi/2} -\cos(x+\pi)+\cos(x+0)\;dx$$

$$2\int_0^{\pi/2} \cos x\,dx = 2\left[\sin x\right]_0^{\pi/2} = 2\left[\sin \pi/2 - \sin 0\right]$$

$$= 2$$

Check: $\displaystyle\int_0^{\pi}\int_0^{\pi/2} \sin(x+y)\,dx\,dy = 2$? (HW)

Q. $\displaystyle\int_0^{2}\int_{-y}^{\sqrt{y}} (1+x+y)\,dx\,dy$, $y \in [0,2]$ $\quad x \in [-y, \sqrt{y}]$ — Dependent Range

$$\int_0^2 dy \cdot \int_{-y}^{\sqrt{y}} (1+x+y)\,dx$$

$$\left[x+\frac{x^2}{2}+xy\right]_{-y}^{\sqrt{y}}$$

$$\left(\sqrt{y}+\frac{y}{2}+y\sqrt{y}\right)-\left(-y+\frac{y^2}{2}-y^2\right)$$

$$\sqrt{y}+\frac{y}{2}+y\sqrt{y}+y-\frac{y^2}{2}+y^2$$

$$\sqrt{y}+y\sqrt{y}+\frac{3y}{2}+\frac{y^2}{2}$$

$$\int_0^2 dy \cdot \left[x+\frac{x^2}{2}+x\cdot y\right]_{-y}^{\sqrt{y}}$$

$$\int_0^2 \sqrt{y}+y\sqrt{y}+\frac{3y}{2}+\frac{y^2}{2}\,dy$$

$$\left[\frac{2}{3}y^{3/2}+\frac{2}{5}y^{5/2}+\frac{3}{4}y^2+\frac{y^3}{6}\right]_0^2$$

$$= \frac{2}{3}(2)^{3/2}+\frac{2}{5}(2)^{5/2}+\frac{3}{4}(2)^2+\frac{(2)^3}{6}$$

Note: Given: $\displaystyle\underbrace{\int_0^{2}\int_{-y}^{\sqrt{y}} (1+x+y)\,dx\,dy}_{\int_0^2 \phi(y)\,dy \,=\, \text{Number}} \neq \underbrace{\int_{-y}^{\sqrt{y}}\int_0^{2} (1+x+y)\,dy\,dx}_{\int_{-y}^{\sqrt{y}} \phi(x)\,dx \,=\, \text{fn of } y}$?

When range of x, y are dependent

$-y$

When range of x, y are dependent, changing the order of integration is not "directly" possible. Hence limit adjustment needs to be done to change the order of integration.

Changing the Order of Integration [for dependent range]:

eg: $$\int_0^1 \int_0^x f(x,y)\, dy\, dx = \int_{y=\_}^{\_} \int_{\_}^{\_} f(x,y)\, dx\, dy$$

$\hookrightarrow$ $y \in [0, x]$, $y = x$

$x \in [0, 1]$.

Alternate: $y \in [0, 1]$

$x \in [y, 1]$

$$\int_0^1 \int_0^x f(x,y)\, dy\, dx = \int_0^1 \int_y^1 f(x,y)\, dx\, dy.$$

Independent plane: