1. Unbiasedness
2. Least variance
3. Consistency.

(i) unbiasedness
(ii) $n \to \infty$
then $var(T) \to 0$

or

$P\{|t_n - \theta| > \varepsilon\} \to 0$ as $n \to \infty$

statistic, parameter

(i) $E(t_n) \to \theta$ (unbiasedness)

and (ii) $var(t_n) \to 0$ as $(n \to \infty)$

4. Efficiency.

asymptotic variance (ie tend to normality for large number of n)

t and t'

$avar(t) \le avar(t')$

then t is efficient estimator of θ.

(5) Sufficiency : $x_1, x_2, \ldots, x_n$ with t

θ as parameter

$$\Rightarrow \left(f(x_1, x_2, \ldots, x_n; \theta) = g(t, \theta) \cdot h(x_1, x_2, \ldots, x_n) \right)$$

Let $x_1, x_2, \ldots x_n$ be independent random observations from a normal population with unknown mean μ and known variance σ^2.

Then the joint probability density function of $x_1, x_2 \ldots x_n$

$$f(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} \left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i - \mu)^2} \right]$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2}$$

We can write $\Sigma(x_i - \mu)^2 = \sum_{i=1}^{n} \left[(x_i - \bar{x}) + (\bar{x} - \mu) \right]^2$

$$= \Sigma(x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

$$\therefore f(x_1, x_2, \ldots, x_n; \theta) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{n}{2\sigma^2}(\bar{x} - \mu)^2} \cdot e^{-\frac{1}{2\sigma^2}\Sigma(x_i - \bar{x})^2}$$

$$f(x_1, \ldots x_n; \theta) = g(\bar{x}; \mu)\, h(x_1, x_2, \ldots, x_n)$$

$\bar{x}$ is a sufficient estimator for μ.

Maximum Likelihood Estimation (MLE)

→ Method of (Point estimation)

Let $x_1, x_2, \ldots, x_n$ be a random sample from a population with p.m.f or p.d.f → $f(x; \theta)$ involving a parameter θ.

For a fixed θ, the function can be written as

$$f(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

This is known as likelihood function of θ and is denoted by $L(\theta)$.

The principle of maximum likelihood suggests us to take that value of θ for which $L(\theta)$ is a maximum.

$$① \quad \left.\frac{d L(\theta)}{d\theta}\right|_{\theta = \hat{\theta}} = 0 \quad \text{and} \quad ② \quad \frac{d^2 L(\theta)}{d\theta^2} < 0$$

Ex: 2.5 .. consider a set of n Bernoullian trials

Ex: Let us consider a set of n Bernoullian trials with p as probability of success in a trial. With the ith trial we associate a variable x_i such that

$$x_i = \begin{cases} 1 & \text{if there is a success} \\ 0 & \text{otherwise} \end{cases}$$

The p.m.f of Bernoulli Distribution is

$$f(x_i; p) = \left\{ p^{x_i} (1-p)^{1-x_i} \right\}; \quad x = 0, 1.$$

So the Likelihood function is

$$L(p) = p^{\sum_{i=1}^{n} x_i} (1-p)^{n - \sum_{i=1}^{n} x_i} \quad \text{——} \; (1)$$

Log on both sides,

$$\log_e L(p) = \left(\sum x_i\right) \log_e p + \left(n - \sum x_i\right) \log_e (1-p)$$

F.O.C $\quad \dfrac{d \log_e L(p)}{dp} = 0$

or, $\quad \dfrac{\sum x_i}{p} - \dfrac{(n - \sum x_i)}{1-p} = 0$

or, $\quad \dfrac{(1-p)\sum x_i - (n - \sum x_i) p}{p(1-p)} = 0$

or, $(1-p)\sum x_i - (n-\sum x_i)p = 0$

or, $\sum x_i - \cancel{p\sum x_i} - np + \cancel{p\sum x_i} = 0$

or, $np = \sum x_i$

or, $\hat{p} = \frac{1}{n}\sum x_i$ = sample mean.

m.l.e.

Suppose $(x_1, x_2, \dots, x_n)$ is a random sample from $N(\mu, \sigma^2)$ where both μ and σ are unknown, we want to find maximum likelihood estimates of μ and σ.

the likelihood function is

$$L(\mu, \sigma) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

then the log-likelihood function can be expressed as,

$$\log_e L(\mu, \sigma) = -n\log_e\sigma - n\log_e\sqrt{2\pi} - \boxed{\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

For estimate of μ, we require F.O.C as,

$$\frac{\partial \log_e L(\mu, \sigma)}{\partial \mu} = 0$$

or, $$0 - 0 - \frac{1}{\cancel{2}\sigma^2} \cancel{2} \sum_{i=1}^{n} (x_i - \mu)(-1) = 0$$

or, $$\sum_{i=1}^{n} (x_i - \mu) = 0$$

or, $$\sum_{i=1}^{n} x_i - n\mu = 0$$

or, $$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \text{Sample mean}$$

m.l.e of μ.

(ii) We have to find m.l.e for σ, F.O.C requires,

$$\frac{\partial \log_e L(\mu, \sigma)}{\partial \sigma} = 0$$

or, $$-\frac{n}{\sigma} - \frac{1}{\cancel{2}\sigma^3}(-\cancel{2}) \sum_{i=1}^{n} (x_i - \mu)^2 = 0$$

or, $$n\sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2$$

or, $$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$, the sample variance

$$\therefore \hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$\therefore \boxed{\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

m.l.e for σ = sample variance