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Real Numbers.

① Nature Numbers.

s_i 1, 2, 3, ..., n

What is the sum of n natural numbers?

$$\Sigma s_i = \frac{n(n+1)}{2}$$

$1^2, 2^2, \ldots, n^2$ (sum of square of n natural no.s!)

$$\Sigma s^2 = \frac{n(n+1)(2n+1)}{6}$$

Basic Properties of the two compositions in N:

① Commutativity of addition and multiplicative

$$a+b = b+a$$

and $ab = ba$ $(a, b \in N)$

② Associativity: $a+(b+c) = (a+b)+c$

$a(bc) = (ab)c$

③ Cancellation: $a + \cancel{c} = b + \cancel{c}$

$a = b$

$a\cancel{c} = b\cancel{c}$

$a = b$

(4) Distributivity: $a(b+c) = ab + ac$

Order structure of set of N of natural numbers

Given any two different natural numbers a, b, we have either $a > b$ or $b > a$

Instead of greater than we can also relate less than or smaller than order relation

$\therefore$ The relation greater than between different natural numbers is known as ORDER RELATION in the set of natural no. and the presence of this relation in N is referred to as "N having an order structure"

PROPERTIES OF ORDER RELATIONS:

(1) Transitivity: if $a > b$

$b > c$

then $a > c$

then $a > c$

(2) Compatibility of the order relation with addition composition.

Let $a > b$

$\Rightarrow a + c > b + c$

(3) Same can be applied with multiplication,

Let $a > b$

$\Rightarrow ac > bc$.

Principle of finite induction:

Let $n \in N$ and let $P(n)$ denote statement pertaining to n. If

(I) $P(1)$ is true i.e. the statement is true for $n = 1$.

(II) $P(n)$ is true $\Rightarrow P(n+1)$ is true.

Inverse Operations and the Corresponding Limitations, Subtraction and Division in N:

Subtraction: Given two numbers (a, c) of (N), does there exist $x \in N$ such that $a + x = c$?

Substitution.

It is easy to see that x, if it exists, is unique.

$$a+x = a+y$$
$$\text{or } x = y$$

Cancellation principle will hold

Also x exists if and only if $c > a$

$a + x = c - a$, $x = c - a$, N, +ve

Division: Given two natural no.s a and c, does there exist a natural no. x such that $ax = c$?

$$x = c/a$$

The no. x, if it exists is unique.

This is a consequence of the cancellation law which states that $ax = ay \Rightarrow x = y$

Also the no. x exists if 'a' is a divisor of 'c'.

The set I or Z of Integers:

$$Z = I = \{\ldots -2, -1, 0, 1, 2, 3, \ldots\}$$

(1) Algebraic structure of the set I of integers, Addition composition in I.

(i) $a+b = b+a$ $[a, b \in I]$. Commutativity.

(ii) $(a+b)+c = a+(b+c)$ $[a, b, c \in I]$ Associativity.

(iii) The no. $0 \in I$ such that $a+0 = a$ $[a \in I]$

additive identity.

(iv) To each $a \in I$ there corresponds another viz $-a \in I$ such that $a + (-a) = 0$

additive inverse.

Inversion of addition

The equation $a + x = b$

$a \in I$, $b \in I$ admits of a unique solution of $x \in I$ such that $a - b \in I$

Multiplication :

(1) $ab = ba$ (2) $(ab)c = a(bc)$

(3) The no. $1 \in I$ is such that $a \cdot 1 = a$ $[a \in I]$

multiplicative identity.

(4) $a(b+c) = ab + bc$

(Distributivity)

Division

If a, b are two non-zero no.s of I, we say that a is a factor of b if there exists $c \in I$ such that $b = ac$.

such that $b = ac$.

ie it will be seen that $b \div a$ is meaningfull iff $a \neq 0$ and a is a factor of b or that a is a divisor of b.

Order Structure of I.

Given any two different numbers $a, b \in I$ we have either $a > b$ or $b > a$.

The 'greater than' relation is transitive

Also $a > b \Rightarrow a + c > b + c$

and $a > b, c > 0 \Rightarrow ac > bc$.

The Set 'Q' of Rational Numbers

Algebraic Structure of Q

As in I, the set Q of rational numbers admits of two composition those are Addition and multiplication.

(1) The addition composition is commutative, associative, admits additive identity ie 0 and each element a admits of an additive inverse ie $(-a)$ for any $a \in Q$

(2) Same as above also admits multiplicative identity 1 and each non zero element

p/q admits of multiplicative inverse is q/p.

Subtraction and Division:

Let a, b be two given rational numbers.

Let's write $a-b = a+(-b)$

we have obtained this by additive inverse $(-b)$ of b

Also if $b \neq 0$, we write

$a \div b = a \times (1/b)$

is obtained on multiplying 'a' with the multiplicative inverse $(1/b)$ of b, $b \neq 0$.

Order Structure of $\mathbb{Q}$.

Given any two different rational numbers a, b, we have either

$a > b$ or $b > a$

(i) $a > b$ and $b > c \Rightarrow a > c$ transitivity.

(ii) $a > b \Rightarrow a+c > b+c$

(iii) $a > b$ and $c > 0 \Rightarrow ac > bc$.

Q Show that there is no rational number whose square is 2.

Let us assume that there exists a rational no. whose square is 2. Let p, q be two integers without a common factor such that the square of rational number is 2 ie

$$\left(\frac{p}{q}\right)^2 = 2$$

$$p^2 = 2q^2$$

Since q is an integer & q^2 is integer

$\therefore 2q^2$ is also integer

As such p must itself be divisible by 2 for otherwise, its square would not be divisible by 2

Let $p = 2n$ when n is an integer,

we have $p^2 = 2q^2$,

$q^2 = 2n^2$

as $p^2 = 4n^2$

so that the integer q^2 is divisible by 2.

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