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Derivatives

$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

functions

1. algebraic $\to x^n$. ✓
2. Trigonometric $\to \sin x / \cos x$.
3. Exponential $\to e^x / a^x$.
4. Logarithmic $\to \log_e x$.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(\underset{A}{x+h}) - \underset{B}{\sin x}}{h}$$

$$= \lim_{h\to 0}\frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h}$$

$$= \boxed{\lim_{h\to 0}\frac{\sin(h/2)}{(h/2)}} \cdot \lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$= 1 \cdot \cos\left(\frac{2x}{2}\right) = \cos x$$

$\sin(-x) = -\sin x$

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\frac{2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{-\sin\left(\frac{h}{2}\right)}{\left(h/2\right)}\sin\left(\frac{2x+h}{2}\right)$$

$$= -\boxed{\lim_{h\to 0}\frac{\sin(h/2)}{(h/2)}} \cdot \lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)$$

$$= -1 \times \sin\left(\frac{2x}{2}\right) = -\sin x$$

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}$$

$$e^{x+h} = e^x \cdot e^h$$

$$= \lim_{h\to 0}\frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x(e^h - 1)}{h}$$

$$= e^x\boxed{\lim_{h\to 0}\frac{e^h - 1}{h}}$$

$$= e^x$$

Taylor Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^h = 1 + h + \frac{h^2}{2!} + \cdots$$

$$e^h - 1 = h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$$

$$\frac{e^h - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots$$

$$\frac{e^h-1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots$$

$$\lim_{h\to 0} \frac{e^h-1}{h} = 1$$

$\log_e x = \ln x.$

$$\frac{d}{dx}(\log_e x) = \lim_{h\to 0} \frac{\log(x+h) - \log x}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h}$$

$$= \frac{1}{x}.$$

$$\log a - \log b = \log\left(\frac{a}{b}\right)$$

$$\log(1+x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\log\left(1+\frac{h}{x}\right) = \frac{h}{x} - \frac{h^3}{3x^3} + \frac{h^5}{5x^5} - \cdots$$

$$\frac{\log\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x} - \frac{h^2}{3x^3} + \frac{h^4}{5x^5} - \cdots$$

$$\lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x}$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \sin x = \cos x.$$

$$\frac{d}{dx} \cos x = -\sin x.$$

$$\frac{d}{dx} e^x = e^x.$$

$$\frac{d}{dx} \log x = \frac{1}{x}.$$

$$\frac{d}{dx} x^{-1} = (-1)x^{-1-1} = -x^{-2}$$

$$= -\frac{1}{x^2}.$$

Rules of derivatives

① $y = f(x) \pm g(x)$ Summation Rule.

$$\frac{dy}{dx} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

② $y = f(x) \cdot g(x)$ Product rule.

$u \cdot v.$

$$\frac{dy}{dx} = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

$u'v + v'u.$

$f'(x) = \frac{d}{dx} f(x)$

$g'(x) = \frac{d}{dx} g(x)$

③ $y = \frac{f(x)}{g(x)} = \frac{u}{v} = u \cdot v^{-1}$ $\frac{1}{v} = v^{-1}$

$$\frac{dy}{dx} = u'v^{-1} + u(v^{-1})'$$

$$= u'\cdot\frac{1}{v} + u\left(-\frac{v'}{v^2}\right)$$

$$= \frac{u'v - v'u}{v^2}.$$

Quotient Rule

④ $y = f[g(x)]$ $g(x) = u.$

$$y = f(u)$$

$$\frac{dy}{du} = \frac{d}{du} f(u)$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}f(u)\cdot\frac{du}{dx}$$
$$= \frac{d}{du}f(u)\cdot\frac{d}{dx}g(x)$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} \quad \frac{u}{v} \quad = \frac{u'v - v'u}{v^2}$$
$$= \frac{\cos x\cos x - (-\sin x)\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$= \underline{\underline{\sec^2 x}}$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) \quad \frac{u}{v} \quad = \frac{u'v - v'u}{v^2} = \frac{0 - (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$
$$= \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x \cdot \sec x$$

$\frac{d}{dx}(a^x) = \frac{d}{dx}(u)$ <u>Method 1</u>

$a^x = u$

$\log a^b = b\log a$

$\log(a^x) = \log u$

$x\log a = \log u$

constant

uv

$$\frac{d}{dx}(cx) = u'v + v'u$$
$$= 0 + c$$
$$= c$$

$$\frac{d}{dx}(cx) = c\,\frac{d}{dx}(x) = c$$

$$\frac{d}{dx}x(\log a) = \frac{d}{dx}(\log u)$$
$$\log a\,\frac{d}{dx}(x) = \frac{d}{du}(\log u)\cdot\frac{du}{dx}$$
$$\log a = \frac{1}{u}\cdot\frac{du}{dx}$$
$$\frac{du}{dx} = u\log a = a^x\log a$$

$$\frac{d}{dx}e^x = e^x$$

$\log a^x = u$

<u>Method 2</u>

$$a^x = e^{[\log(a^x)]}$$

$$\log e^x = x = e^{\log x}$$
$$\log x = \log x\cdot\log e = \log x$$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}e^{(\log a^x)}$$

$\log a^x = x\log a$

$$= \frac{d}{dx}e^u = \frac{d}{du}e^u\cdot\frac{du}{dx} = e^u\cdot\frac{d}{dx}(\log a^x)$$
$$= e^u\cdot\frac{d}{dx}[x\log a] = e^u\log a = \underline{a^x\log a}$$

$\boxed{nx = u}$

$$\frac{d}{dx}\sin(nx) = \frac{d}{dx}\sin u = \frac{d}{du}\sin u \cdot \frac{du}{dx} = \cos u \cdot \frac{d}{dx}(nx)$$

$$= \cos(nx)\; n\cdot\frac{d}{dx}(x) = n\cdot\cos(nx)$$

$$\frac{d}{dx}\cos(nx) = -n\sin(nx)$$

$nx = u$

$$\frac{d}{dx}\left(e^{nx}\right) = \frac{d}{dx}e^{u} = \frac{d}{du}e^{u}\cdot\frac{du}{dx} = e^{u}\cdot\frac{d}{dx}(nx) = e^{nx}\cdot n$$

$$= ne^{nx}$$

$$\frac{d}{dx}\log(nx) = \frac{1}{nx}\cdot n = \frac{1}{x}.$$

Function of functions

$y = \sin(x^2)$

$x^2 = u$

$$\frac{d}{dx}x^2 = 2x.$$

$$\frac{dy}{dx} = \frac{d}{dx}\sin u = \frac{d}{du}\sin u\cdot\frac{du}{dx} = \cos u\cdot\frac{d}{dx}(x^2)$$

$$= 2x\cos(x^2)$$

$y = \log(\sin x)$, u.

$$\frac{dy}{dx} = \frac{d}{dx}\log u = \frac{d}{du}\log u\cdot\frac{du}{dx}$$

$$= \frac{1}{u}\cdot\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cdot\cos x = \cot x$$

$y = x^{(e^x)}$

$$\log y = \log x^{(e^x)} = e^x\cdot\log x.$$

$u\cdot v$.

$u = e^x \quad u' = e^x.$

$v = \log x \quad v' = \frac{1}{x}.$

$$\frac{d}{dx}(\log y) = u'v + v'u = e^x\log x + \frac{1}{x}e^x = e^x\left[\log x + \frac{1}{x}\right]$$

$$\frac{d}{dy}\log y\cdot\frac{dy}{dx} = e^x\left[\log x + \frac{1}{x}\right]$$

$$\frac{1}{y}\frac{dy}{dx} = e^x\left[\log x + \frac{1}{x}\right]$$

$$\frac{dy}{dx} = y\cdot e^x\left[\log x + \frac{1}{x}\right] = x^{e^x}\cdot e^x\left[\log x + \frac{1}{x}\right]$$

$y = x^x$

$\log y = x \log x.$

$\frac{d}{dx} \log y = \log x + x \cdot \frac{1}{x} = \log x + 1$

$\frac{d}{dy} \log y \cdot \frac{dy}{dx} = \log x + 1$

$\frac{1}{y} \cdot \frac{dy}{dx} = \log x + 1$

$\frac{dy}{dx} = y(\log x + 1) = x^x(\log x + 1)$