Binomial Theorem.

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}\cdot b + {}^nC_2\, a^{n-2}\, b^2 + \cdots + {}^nC_n \cdot b^n \quad \text{--- (i)}$$

Note:

(i) ${}^nC_r = {}^nC_{n-r}$.

(ii) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$.

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

(iv) ${}^nC_p = {}^nC_q \Rightarrow$ Either $p = q$ or $p+q = n$.

In (i) Put $a=1$, $b=x$:-

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \cdots + {}^nC_n \cdot x^n \quad \text{---- (ii)}$$

In (ii), Put $x=1$:-

$$\boxed{2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n} \quad \text{----- (iii)}$$

In (ii), Put $x=-1$:-

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots (-1)^n\, {}^nC_n.$$

$$\Rightarrow \left({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots\right) = \left({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots\right) \quad \text{--- (iv)}$$

[Sum of even coeffs] = [Sum of odd coeffs].

Put (iv) in (iii) :-

$$2^n = \underbrace{\left({}^nC_0 + {}^nC_2 + \cdots\right)}_{x} + \underbrace{\left({}^nC_1 + {}^nC_3 + \cdots\right)}_{x}$$

$$2^n = 2x \Rightarrow \boxed{x = 2^{n-1}}$$

$$\Rightarrow \left({}^nC_0 + {}^nC_2 + \cdots\right) = \left({}^nC \ldots\right) \ldots {}^{n-1}$$

$$\Rightarrow \boxed{\left({}^nC_0 + {}^nC_2 + \cdots\right) = \left({}^nC_1 + {}^nC_3 + \cdots\right) = 2^{n-1}} \text{ ---- (v)}$$

Differentiate eqn (ii) :-

(ii) : $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots + {}^nC_n x^n$

Diff w.r.t 'x' :-

$$n(1+x)^{n-1} = {}^nC_1 + 2x\,{}^nC_2 + 3x^2\,{}^nC_3 + \cdots + nx^{n-1}\,{}^nC_n .$$

Put $x = 1$:-

$$\boxed{n \cdot 2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \cdots + n\,{}^nC_n} \text{ --- (vi)}$$

Integrate eqn (ii) :-

(ii) $\Rightarrow$ $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n .$

$$\frac{(1+x)^{n+1}}{n+1} = {}^nC_0\, x + {}^nC_1 \frac{x^2}{2} + {}^nC_2 \frac{x^3}{3} + \cdots + {}^nC_n \frac{x^{n+1}}{n+1} + k .$$

To evaluate k, put $x = 0 \Rightarrow \frac{1}{n+1} = k .$

$$\frac{(1+x)^{n+1}}{n+1} = {}^nC_0\, x + {}^nC_1 \frac{x^2}{2} + {}^nC_2 \cdot \frac{x^3}{3} + \cdots + {}^nC_n \cdot \frac{x^{n+1}}{n+1} + \frac{1}{n+1} .$$

Put $x = 1$:-

$$\frac{2^{n+1}}{n+1} = {}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \cdots + \frac{{}^nC_n}{n+1} + \boxed{\frac{1}{n+1}}$$

$$\boxed{\frac{2^{n+1} - 1}{n+1} = {}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \cdots + \frac{{}^nC_n}{n+1}} \text{ ---- (vii)}$$

Q. $^nC_0 + 2\,^nC_1 + 3\,^nC_2 + \cdots + (n+1)\,^nC_n = ?$

(a) 2^{n+2} $\quad$ (b) $2^n + n\,2^{n-1}$ $\quad$ (c) $2^n - n\,2^{n-1}$ $\quad$ (d) $2^n + n\cdot 2^n$.

From Diff series:

$$1\cdot{}^nC_1 + 2\,^nC_2 + 3\,^nC_3 + \cdots + n\,^nC_n = n\cdot 2^{n-1}.$$

$x = k$

$x + a - a = k$

$$^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = 2^n.$$

$$^nC_0 + 2\cdot{}^nC_1 + 3\,^nC_2 + \cdots + (n+1)\,^nC_n = 2^n + n\,2^{n-1} \quad (b)$$

Q. $\dfrac{^{30}C_1}{2} + \dfrac{^{30}C_3}{4} + \dfrac{^{30}C_5}{6} + \cdots + \dfrac{^{30}C_{29}}{30} = ?$

(a) $\dfrac{2^{31}}{30}$ $\quad$ (b) $\dfrac{2^{30}}{31}$ $\quad$ (c) $\dfrac{2^{31}-1}{31}$ $\quad$ (d) $\dfrac{2^{30}-1}{31}$

$$^nC_0\,x + {}^nC_1\cdot\frac{x^2}{2} + {}^nC_2\cdot\frac{x^3}{3} + \cdots + {}^nC_n\cdot\frac{x^{n+1}}{n+1} = \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1}$$

Put $x = 1$: $\quad ^nC_0 + \dfrac{^nC_1}{2} + \dfrac{^nC_2}{3} + \cdots + \dfrac{^nC_n}{(n+1)} = \dfrac{2^{n+1}-1}{(n+1)}$

Put $x = -1$: $\quad -{}^nC_0 + \dfrac{^nC_1}{2} - \dfrac{^nC_2}{3} + \cdots\cdots = -\dfrac{1}{n+1}$

$n = 30 \Rightarrow$ $\quad ^{30}C_0 + \dfrac{^{30}C_1}{2} + \dfrac{^{30}C_2}{3} + \cdots + \dfrac{^{30}C_{29}}{30} + \dfrac{^{30}C_{30}}{31} = \dfrac{2^{31}-1}{31}.$

$$-{}^{30}C_0 + \frac{^{30}C_1}{2} - \frac{^{30}C_2}{3} + \cdots + \frac{^{30}C_{29}}{30} - \frac{^{30}C_{30}}{31} = -\frac{1}{31}$$

$$\overline{\hspace{10em}}\quad 31$$

Add: $2\left[\frac{{}^{30}C_1}{2}+\frac{{}^{30}C_3}{4}+\cdots+\frac{{}^{30}C_{29}}{30}\right]=\frac{2^{31}-1}{31}-\frac{1}{31}=\frac{2^{31}-2}{31}.$

$$\frac{{}^{30}C_1}{2}+\frac{{}^{30}C_3}{4}+\cdots+\frac{{}^{30}C_{29}}{30}=\frac{1}{2}\left[\frac{2^{31}-2}{31}\right]=\frac{2^{30}-1}{31}$$

HW

Q. Find the coeff of x^{950} in the expansion of:

$$(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\cdots+x^{1000}.$$

(a) ${}^{1000}C_{950}$ (b) ${}^{1001}C_{950}$ (c) ${}^{1001}C_{949}$ (d) None.