22 July 2023 11:50

Trigonometry.

Let x,y, z be real numbers with $x \geqslant y \geqslant z \geqslant \frac{\pi}{8}$ such that $\boxed{x+y+z=\frac{\pi}{2}}$ and let $P = \cos x . \sin y . \cos z$ then which is/are CORRECT?

(A) Minimum value of P is $\frac{1}{8}$ (B) Minimum value of P is $\frac{1}{4}$

(C) Maximum value of P is $\frac{2+\sqrt{3}}{4}$ (D) Maximum value of P is $\frac{\sqrt{2}+1}{4\sqrt{2}}$

$P = \cos x \sin y \cos z$

$x \geqslant y \geqslant z \geqslant \frac{\pi}{8}.$

$y = z = \pi/8$

$x = \frac{\pi}{2} - 2\times\frac{\pi}{8} = \frac{\pi}{4}.$

$\boxed{\frac{\pi}{4}} \geqslant x \geqslant y \geqslant z \geqslant \frac{\pi}{8}.$

$0 \leftarrow \pi/2$

$1 \leftarrow 0$

$P = \frac{1}{2}\cos x\,[2\sin y \cos z].$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$= \frac{1}{2}\cos x\,[\sin(y+z) + \sin(y-z)].$

$y+z = \frac{\pi}{2} - x.$

$= \frac{1}{2}\cos x\left[\sin\left(\frac{\pi}{2}-x\right) + \sin(y-z)\right]$

$y \geqslant z. \Rightarrow y-z \geqslant 0$

$= \frac{1}{2}\cos x\,[\cos x + \sin(y-z)].$

$\sin(y-z) \geqslant 0$

$x \geqslant y$

$P \geqslant \frac{1}{2}\cos^2 x.$ $P_{min} = \frac{1}{2}\cos^2 x.$

$\cos x$ is min when x is maximum.

$y - x \leqslant 0.$

$= \frac{1}{2}\cdot\cos^2\frac{\pi}{4} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$\sin(y-x) \leqslant 0$

$\sin(y-x) = -\sin(x-y)$

$P = \frac{1}{2}\cdot[2\cos x \sin y]\cos z = \frac{1}{2}[\sin(y+x) + \sin(y-x)]\cos z.$

$= \frac{1}{2}[\sin(x+y) - \sin(x-y)]\cos z.$

$x+y = \frac{\pi}{2} - z$

$\sin(x+y) = \cos z.$

$= \frac{1}{2}[\cos z - \sin(x-y)]\cos z.$

$\boxed{\cos 2x = 2\cos^2 x - 1}$

$P \leq \frac{1}{2}\cos^2 z.$ $P_{max} = \frac{1}{2}\cos^2 z.$ $z = \frac{\pi}{8}.$

$P_{max} = \frac{1}{2}\cdot\frac{1}{2}\cdot 2\cos^2 z = \frac{1}{4}[\cos 2z + 1] = \frac{1}{4}\left[\frac{1}{\sqrt{2}} + 1\right]$

$= \frac{1}{4}\cdot\frac{\sqrt{2}+1}{\sqrt{2}} = \frac{\sqrt{2}+1}{4\sqrt{2}}$

...ositive integer...

The value of $\frac{(1+\sqrt{3}\tan 1^\circ)(1+\sqrt{3}\tan 2^\circ)(\tan 1^\circ + \tan 59^\circ)(\tan 2^\circ + \tan 58^\circ)}{(1+\tan^2 1^\circ)(1+\tan^2 2^\circ)}$ is

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$

$\frac{\left(1+\sqrt{3}\frac{\sin 1}{\cos 1}\right)\left(1+\sqrt{3}\frac{\sin 2}{\cos 2}\right)\left(\frac{\sin 60}{\cos 1\cos 59}\right)\left(\frac{\sin 60}{\cos 2\cos 58}\right)}{\sec^2 1\ \sec^2 2}$

$\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}$

$= \frac{\sin(A+B)}{\cos A\cos B}.$

$= \frac{(\cos 1 + \sqrt{3}\sin 1)(\cos 2 + \sqrt{3}\sin 2)\ (3/4)}{\cancel{\cos^2 1}\,\cancel{\cos^2 2}\,\cos 59\cos 58\,\cancel{\sec^2 1}\,\cancel{\sec^2 2}} = \frac{3}{4}\,\frac{(\cos 1 + \sqrt{3}\sin 1)(\cos 2 + \sqrt{3}\sin 2)}{\cos 59\cos 58}.$

$= 3\,\frac{\cancel{\left(\frac{1}{2}\cos 1 + \frac{\sqrt{3}}{2}\sin 1\right)}\cancel{\left(\frac{1}{2}\cos 2 + \frac{\sqrt{3}}{2}\sin 2\right)}}{\cancel{\cos 59}\,\cancel{\cos 58}}$

$\frac{1}{2}\cos 1 + \frac{\sqrt{3}}{2}\sin 1 = \cos 60\cos 1 + \sin 60 \sin 1$

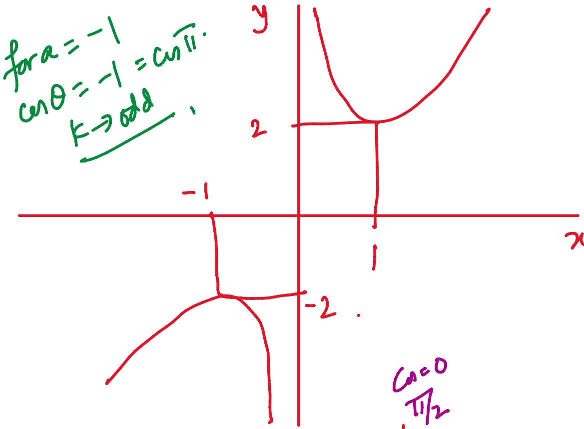
$= \cos(60-1) = \cos 59$

$= 3.$

6. If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$ and $\cos 3\theta = \frac{1}{2}\left(a^k+\frac{1}{a^k}\right)$ then number of natural numbers 'k' less than 50 is (given $a \in R$)

$k = 1$ to 49.

for $a = -1$
$\cos\theta = -1 = \cos\pi$.
$k \to$ odd



$x + \frac{1}{x}$.

$a + \frac{1}{a} \geq 2$. $\qquad a + \frac{1}{a} \leq -2$.

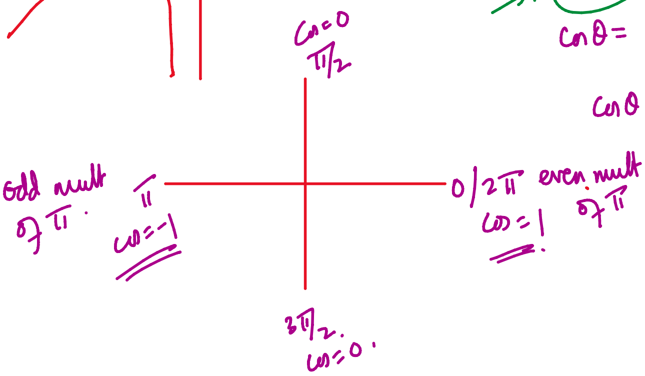
$a+\frac{1}{a} > 2 \Rightarrow \frac{1}{2}\left(a+\frac{1}{a}\right) > 1 \Rightarrow \cos\theta > 1$ ✗

$a = 1 \leftarrow \boxed{a+\frac{1}{a} = 2} \Rightarrow \frac{1}{2}\left(a+\frac{1}{a}\right) = 1 \Rightarrow \boxed{\cos\theta = 1}$

$a = -1 \leftarrow \boxed{a+\frac{1}{a} = -2} \Rightarrow \boxed{\cos\theta = -1}$

$\cos\theta = \cos 0 \qquad \boxed{\theta = 2n\pi}$

$\cos\theta = \cos\pi$. $\qquad \boxed{\theta = (2n+1)\pi} \qquad \longrightarrow \boxed{\theta = n\pi}$



$\cos 3\theta = \cos(6n\pi)$ or $\cos[3(2n+1)\pi]$
$= 1$ or -1

$\frac{1}{2}\left(a^k + \frac{1}{a^k}\right) = \pm 1$

$a^k + \frac{1}{a^k} = \pm 2 \Rightarrow a^k = 1$ or -1

no of values of $k = 49$.

$a = 1 \quad a^k = 1$ for all values of k.

$a = -1 \quad a^k = 1$ for even values of k.
$\quad = -1$ " odd " " k.

If $2\tan^2\theta_1\ \tan^2\theta_2\tan^2\theta_3 + \tan^2\theta_1\tan^2\theta_2 + \tan^2\theta_2\tan^2\theta_3 + \tan^2\theta_3\tan^2\theta_1 = 1$ then which of the follow relations hold good?

(A) $\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 1$ (B) $\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3 = 1$

(C) $\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 2$ (D) $\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3 = -1$

tanA + tanB ⇓ convert to sin/cos → sin(A+B)/cosAcosB

multiply both sides by $\cot^2\theta_1 \cot^2\theta_2 \cot^2\theta_3$.

$2 + \cot^2\theta_3 + \cot^2\theta_2 + \cot^2\theta_1 = \cot^2\theta_1\cot^2\theta_2\cot^2\theta_3$.

$1+\cot^2\theta = \csc^2\theta$.

$\cot^2\theta = \csc^2\theta - 1$

$\cancel{2} + \csc^2\theta_3 - \cancel{1} + \csc^2\theta_2 - \cancel{1} + \csc^2\theta_1 - 1 = (\csc^2\theta_1 - 1)(\csc^2\theta_2 - 1)(\csc^2\theta_3 - 1)$

$\cancel{\csc^2\theta_1} + \cancel{\csc^2\theta_2} + \cancel{\csc^2\theta_3} - \cancel{1} = \csc^2\theta_1\csc^2\theta_2\csc^2\theta_3 + \cancel{\csc^2\theta_1} + \cancel{\csc^2\theta_2} + \cancel{\csc^2\theta_3}$
$- \csc^2\theta_1\csc^2\theta_2 - \csc^2\theta_2\csc^2\theta_3 - \csc^2\theta_3\csc^2\theta_1 - \cancel{1}$

$0 = 1 - \sin^2\theta_1 - \sin^2\theta_2 - \sin^2\theta_3$.

$\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 1$

$2\sin^2\theta_1 + 2\sin^2\theta_2 + 2\sin^2\theta_3 = 2$

$\cancel{1} - \cos 2\theta_1 + \cancel{1} - \cos 2\theta_2 + 1 - \cos 2\theta_3 = \cancel{2}$

$\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3 = 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

If $\dfrac{\prod_{n=1}^{89}(\sin n^\circ + \cos n^\circ)}{\prod_{n=1}^{45}\cos n^\circ \prod_{n=46}^{89}\sin n^\circ} = 2^k$ then value of $\left[\dfrac{K}{20}\right]$ is 2. (where [.] denotes greatest integer function)

$K/20 = 2.25$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

n=1. $\sin 1^\circ + \cos 1^\circ = \sin 1^\circ + \sin 89^\circ = 2\sin 45^\circ \cos 44^\circ = 2 \cdot \frac{1}{\sqrt{2}}\cos 44 = \sqrt{2}\cos 44^\circ$.

n=2 $\sin 2^\circ + \cos 2^\circ = \sqrt{2}\cos 43^\circ$

n=3 $\sin 3^\circ + \cos 3^\circ = \sqrt{2}\cos 42^\circ$.

⋮

n=44 $\sin 44 + \cos 44 = \sqrt{2}\cos 1^\circ$.

n=45° $\sin 45 + \cos 45 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$.

n=46. $\sin 46 + \cos 46 = \sqrt{2}\cos 1^\circ$

n=47 $\sin 47 + \cos 47 = \sqrt{2}\cos 2^\circ$.

⋮

n=89 $\sin 89 + \cos 89 = \sqrt{2}\cos 44$.

$\prod_{n=46}^{89}\sin n^\circ = \prod_{n=1}^{44}\cos n^\circ$

$\underline{n=44}$

$\underline{n=89}$. $\sin 89 + \cos 89 = \sqrt{2}\cos 44.$

$$\prod_{n=46}^{89} \sin n^\circ = \prod_{n=1}^{44} \cos n^\circ$$

$\underline{\underline{K=45}}$

$\frac{K}{20} = \frac{22.5}{10} = 2.25$

$\text{Num} = (\sqrt{2})^{88} \cos^2 1^\circ \cos^2 2 \cos^2 3 \dots \cos^2 44 \cdot (\sqrt{2})$

$\text{Den} = \cos^2 1^\circ \cos^2 2 \cos^2 3 \dots \cos^2 44 \cdot \left(\frac{1}{\sqrt{2}}\right)$

$\text{Exp} = (\sqrt{2})^{90} = 2^{45}.$