Let

$$f_n(x) = (2 + (-2)^n)\,x^2 + (n+3)\,x + n^2$$

where n is a positive integer and x is any real number.

(i) Write down $f_3(x)$.

Find the maximum value of $f_3(x)$.

For what values of n does $f_n(x)$ have a maximum value (as x varies)?

[Note you are not being asked to calculate the value of this maximum.]

(ii) Write down $f_1(x)$.

Calculate $f_1(f_1(x))$ and $f_1(f_1(f_1(x)))$.

Find an expression, simplified as much as possible, for

$$f_1(f_1(f_1(\cdots f_1(x))))$$

where f_1 is applied k times. [Here k is a positive integer.]

(iii) Write down $f_2(x)$.

The function

$$f_2(f_2(f_2(\cdots f_2(x)))),$$

where f_2 is applied k times, is a polynomial in x. What is the degree of this polynomial?

$f_3(x) = -6x^2 + 6x + 9$

$x = 1/2 \quad f_3(x)_{max}$ ✓

$n \to$ odd.

$f_1(x) = 4x + 1$

$f_1(f_1(x)) = 4(4x+1) + 1 = 4^2 x + (4+1)$

$f_1(f_1(f_1(x))) = 4(4^2 x + (4+1)) + 1$

$= 4^3 x + 4(4) + 4 + 1$

$= 4^3 x + 4^2 + 4 + 1$

$f_1^k(x) = 4^k x + (1 + 4 + \cdots + 4^{k-1})$

$= 4^k x + \frac{4^k - 1}{3}$.

$f_2(x) = 6x^2 + 5x + 4$ $\quad 2^k$.

$f_2(f_2(x)) = 6(6x^2 + 5x + 4)^2 + 5(6x^2 + 5x + 4) + 4$

Let

$$I(c) = \int_0^1 \left((x-c)^2 + c^2\right)\,dx$$

+ve +ve.

$\boxed{\text{where } c \text{ is a real number.}}$

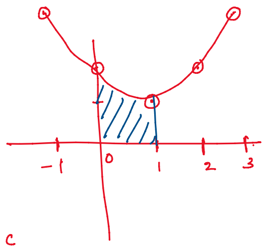
(i) Sketch $y = (x-1)^2 + 1$ for the values $-1 \leqslant x \leqslant 3$ on the axes below and show on your graph the area represented by the integral $I(1)$.

(ii) Without explicitly calculating $I(c)$, explain why $I(c) \geqslant 0$ for any value of c.

(iii) Calculate $I(c)$.

(iv) What is the minimum value of $I(c)$ (as c varies)? $= 0$.

(v) What is the maximum value of $I(\sin\theta)$ as θ varies?



$I(c) = \int_0^1 (x^2 - 2cx + 2c^2)\,dx$

$\left.\frac{x^3}{3} - 2c\frac{x^2}{2} + 2c^2 x\right|_0^1$

$= \frac{1}{3} - c + 2c = \boxed{c + \frac{1}{3}}$

$f(x) = (x-c)^2 + c^2$

for any value of c

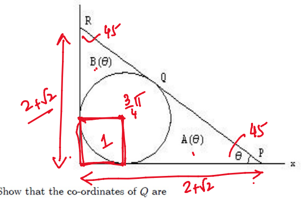
both terms are $\geqslant 0 \Rightarrow f(x) \geqslant 0$.

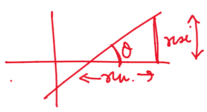
$I(c)$ is the area under $f(x)dx$ where x varies

from $[0,1] \Rightarrow I(c) \geqslant 0$

$I(\sin\theta) = \sin\theta + \frac{1}{3}$.

In the diagram below is sketched the circle with centre $(1,1)$ and radius 1 and a line L. The line L is tangential to the circle at Q; further L meets the y-axis at R and the x-axis at P in such a way that the angle OPQ equals θ where $0 < \theta < \pi/2$.





$m = \frac{\Delta y}{\Delta x}$.

$= \tan\theta$

(i) Show that the co-ordinates of Q are

$$(1 + \sin\theta, 1 + \cos\theta),$$

and that the gradient of PQR is $-\tan\theta$.

$Q = [(1+\sin\theta), (1+\cos\theta)]$

Slope of a line = tan(angle with the +ve direction of x axis)

$= \tan(180 - \theta) = -\tan\theta$.

$x = 1 + \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}$

equ of a line through a pt.

$y - (1 + \cos\theta) = (-\tan\theta)(x - 1 - \sin\theta)$

$= -\tan\theta\, x + \tan\theta + \tan\theta$

(i) Show that the co-ordinates of Q are $2+\sqrt{2}$

$$(1+\sin\theta, 1+\cos\theta),$$

and that the gradient of PQR is $-\tan\theta$.

Write down the equation of the line PQR and so find the co-ordinates of P.

(ii) The region bounded by the circle, the x-axis and PQ has area $A(\theta)$; the region bounded by the circle, the y-axis and QR has area $B(\theta)$. (See diagram.)

Explain why

$$A(\theta) = B(\pi/2 - \theta)$$

for any θ.

Calculate $A(\pi/4)$.

(iii) Show that

$$\boxed{A\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3}.}$$

$x = 1 + \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}$

$= 2 + \sqrt{2}$

$2A\left(\frac{\pi}{4}\right) + \frac{3}{4}\pi + 1$

$= \frac{1}{2}(2+\sqrt{2})^2$

$= \frac{1}{2}(6+4\sqrt{2})$ $x =$

$2A\left(\frac{\pi}{4}\right) + \frac{3\pi}{4} + 1 = 3 + 2\sqrt{2}$

equ of a line through a pt.

$y - (1+\cos\theta) = (-\tan\theta)(x - 1 - \sin\theta)$

$y - 1 - \cos\theta = -\tan\theta\, x + \tan\theta + \tan\theta \sin\theta$

$y = -\tan\theta\, x + \tan\theta + \tan\theta\sin\theta + \cos\theta + 1$

$y = 0$

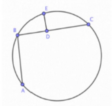
$\tan\theta\, x = \tan\theta(1+\sin\theta) + 1 + \cos\theta.$

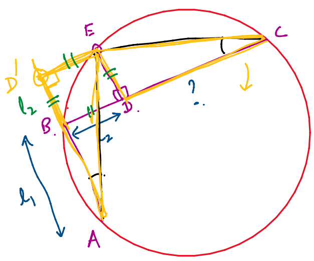
$x = 1 + \sin\theta + \cot\theta + \cot\theta\cos\theta.$

$2A\left(\frac{\pi}{4}\right) = 2 + 2\sqrt{2} - \frac{3}{4}\pi.$

$A\left(\frac{\pi}{4}\right) = 1 + \sqrt{2} - \frac{3}{8}\pi$

In the figure below, E is the midpoint of the arc $ABEC$ and the segment ED is perpendicular to the chord BC at D. If the length of the chord AB is l_1, and that of the segment BD is l_2, determine the length of DC in terms of l_1, l_2.





$AE = EC$

$\angle EAB = \angle ECB$

$= \angle ECD$

$AD' = CD$

$l_1 + l_2 = CD$

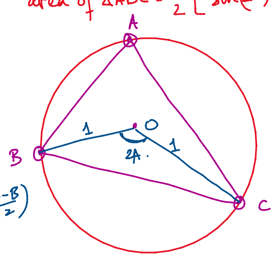
Let A, B and C be three points on a circle of radius 1.

(a) Show that the area of the triangle ABC equals $\frac{1}{2}(\sin(2\angle ABC) + \sin(2\angle BCA) + \sin(2\angle CAB))$

(b) Suppose that the magnitude of $\angle ABC$ is fixed. Then show that the area of the triangle ABC is maximized when $\angle BCA = \angle CAB$

(c) Hence or otherwise, show that the area of the triangle ABC is maximum when the triangle is equilateral.

area of $\Delta ABC = \frac{1}{2}[\sin(2A) + \sin(2B) + \sin(2C)]$



$\sin A + \sin B$

$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\Delta = \frac{1}{2}[\sin 2A + \sin 2C]$

area of $\Delta ABC = \Delta AOC + \Delta BOC + \Delta AOB.$

$\Delta BOC = \frac{1}{2} \cdot 1 \times 1 \times \sin(2A)$

$= \frac{1}{2}\sin(2A)$

$\Delta = \frac{1}{2}[\sin 2A + \sin 2B + \sin 2C] = \frac{1}{2}[2\sin(A+B)\cos(A-B) + 2\sin C\cos C]$

$0^\circ < A, C < 180^\circ$

$0 < \sin A, \sin C < 1$

$= \sin C\cos(A-B) + \sin C\cos[180 - (A+B)]$

$= \sin C[\cos(A-B) - \cos(A+B)]$

$A + B = 180 - C$

$(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$

$\frac{a^2+b^2}{2} \geq ab$

$GM(a,b)$

$= \sqrt{ab}$

$$\Delta = \frac{1}{2}\left[\sin 2A + \sin 2C\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\sin(A+C)\cos(A-C)\right]$$

$$= \sin B \cos(A-C)$$

$$= 1 \Rightarrow A - C = 0 \Rightarrow \underline{A = C}$$

$0^\circ < A, C < 180^\circ$

$0 < \sin A, \sin C < 1$

$A + B + C = 180^\circ$ $\quad \underline{A + B = 180 - C}$

$A + C = 180 - B.$ $\quad C = 180 - (A+B)$

$$= \sin C \cos(A-B) + \sin C \cos[180 - (A+B)]$$

$$= \sin C[\cos(A-B) - \cos(A+B)]$$

$$= \sin C \cdot 2\sin A \sin B.$$

$$= \underline{2\sin A \sin B \sin C}.$$

$\underline{AM \geq GM}$

$$\frac{\sin A + \sin B + \sin C}{3} \geq \sqrt[3]{\sin A \sin B \sin C}$$

$\underline{A = B = C}$.

$a^2 + b^2 \geq 2ab$

$\frac{a^2+b^2}{2} \geq ab$

$\frac{a^2+b^2}{2} \geq \sqrt{a^2b^2}$.

$AM(a^2, b^2) \geq GM(a^2, b^2)$

A man standing at a point O finds that a balloon at a height h metres due east of him has an angle of elevation 60°. He walks due north while the balloon moves north-west (45° west of north) remaining at the same height. After he has walked 100 metres the balloon is vertically above him. Then the value of h in metres is

(A) 50 ;
(B) $50\sqrt{3}$
(C) $100\sqrt{3}$;
(D) $\frac{100}{\sqrt{3}}$

Let $f(x) = e^x$

$$g(x) = \begin{cases} x^2 & \text{if } x < 1/2 \\ x - \frac{1}{4} & \text{if } x \geq 1/2 \end{cases}$$

and $h(x) = f(g(x))$. The derivative of h at $x = 1/2$

(A) is e;
(B) is $e^{1/2}$;
(C) is $e^{1/4}$;
(D) does not exist.

Let ABC be a right angled triangle with $BC = 3$ and $AC = 4$. Let D be a point on the hypotenuse AB such that $\angle BCD = 30^\circ$. The length of CD is

(A) $\frac{24}{3+4\sqrt{3}}$;
(B) $\frac{3\sqrt{3}}{2}$
(C) $6\sqrt{3} - 8$
(D) $\frac{25}{12}$.

Let a be a positive number. Then

$\lim_{n\to\infty}\left[\frac{1}{a+n}+\frac{1}{2a+n}+\ldots+\frac{1}{an+n}\right]$ equals

(A) 0
(B) $\log_e(1+a)$

(C) $\frac{1}{a}\log_e(1+a)$
(D) none of these expressions.

Let $f(n)$ be a function defined, for any integer $n \geqslant 0$, as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ (f(n/2))^2 & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 2f(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(i) What is the value of $f(5)$?

The *recursion depth* of $f(n)$ is defined to be the number of other integers m such that the value of $f(m)$ is calculated whilst computing the value of $f(n)$. For example, the recursion depth of $f(4)$ is 3, because the values of $f(2), f(1)$, and $f(0)$ need to be calculated on the way to computing the value of $f(4)$.

(ii) What is the recursion depth of $f(5)$?

Now let $g(n)$ be a function, defined for all integers $n \geqslant 0$, as follows:

$$g(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 + g(n/2) & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 1 + g(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(iii) What is $g(5)$?

(iv) What is $g(2^k)$, where $k \geqslant 0$ is an integer? Briefly explain your answer.

(v) What is $g(2^l + 2^k)$ where $l > k \geqslant 0$ are integers? Briefly explain your answer.

(vi) Explain briefly why the value of $g(n)$ is equal to the recursion depth of $f(n)$.