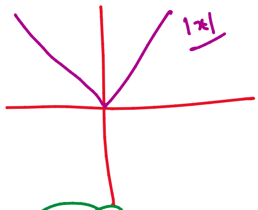
08 August 2023 15:05

A. How many real solutions x are there to the equation $x|x| + 1 = 3|x|$?

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

[Note that $|x|$ is equal to x if $x \geqslant 0$, and equal to $-x$ otherwise.]

$y = |x| \to y = x, \; x \geq 0$
$= -x, \; x < 0$



for $x \geq 0$

$x^2 + 1 = 3x$

$x^2 - 3x + 1 = 0$

$D = b^2 - 4ac = 9 - 4 = 5 > 0$

2 solns

for $x < 0$.

$x(-x) + 1 = 3(-x)$

$-x^2 + 1 = -3x$

$x^2 - 3x - 1 = 0.$

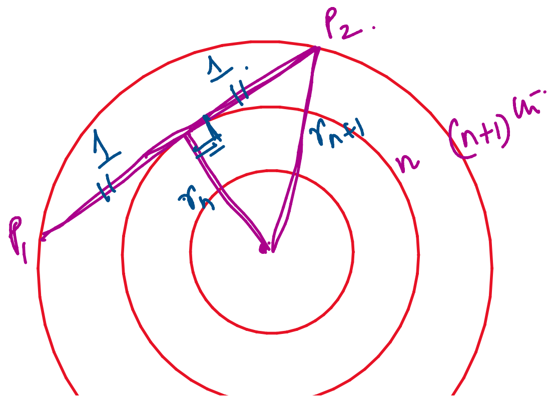
$D = 9 + 4 = 13 > 0$

2 solns.

B. One hundred circles all share the same centre, and they are named C_1, C_2, C_3, and so on up to C_{100}. For each whole number n between 1 and 99 inclusive, a tangent to circle C_n crosses circle C_{n+1} at two points that are separated by a distance of 2. Given that C_1 has radius 1, it follows that the radius of C_{100} is

(a) 1, (b) 2, (c) $\sqrt{10}$, (d) 10, (e) 100.

Proof of tangents & chords



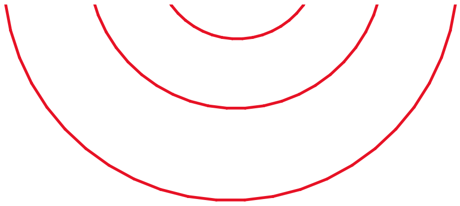
$r_{n+1}^2 = r_n^2 + 1^2$

$r_{n+1}^2 = r_n^2 + 1$

$n = 1.$ $\quad r_2^2 = r_1^2 + 1$

$r_2^2 = 1 + 1 = 2.$

$r_1 = 1$



$r_1 = 1$

$r_2^2 = 1+1 = 2.$

$r_3^2 = r_2^2 + 1 = 3.$

$r_4^2 = r_3^2 + 1 = 4$

$r_n^2 = n.$

$r_{100}^2 = 100$

$r_{100} = \sqrt{100} = 10.$

C. The equation $x^2 - 4kx + y^2 - 4y + 8 = k^3 - k$ is the equation of a circle

(a) for all real values of k.

(b) if and only if either $-4 < k < -1$ or $k > 1$. ✓

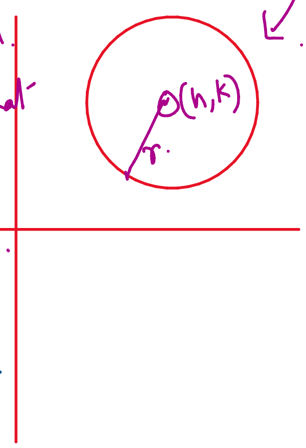
(c) if and only if $k > 1$.

(d) if and only if $k < -1$.

(e) if and only if either $-1 < k < 0$ or $k > 1$.

generic equation of a circle.

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$



$x^2 - 4kx + y^2 - 4y + 8 = k^3 - k$. rearranged in this format.

$$[x^2 - 2(2k)x + (2k)^2] - (2k)^2 + [y^2 - 2(2)y + (2)^2] - (2)^2 + 8 = k^3 - k.$$

$$(x-2k)^2 + (y-2)^2 - 4k^2 - 4 + 8 = k^3 - k.$$

$$(x-2k)^2 + (y-2)^2 = [k^3 + 4k^2 - k - 4]$$

$r^2 > 0.$

$k^3 + 4k^2 - k - 4 > 0.$

$k^2(k+4) - 1(k+4) > 0.$

$(k+4)(k^2-1) > 0.$

$(k+4)(k+1)(k-1) > 0.$

Critical points points where the expression $= 0$

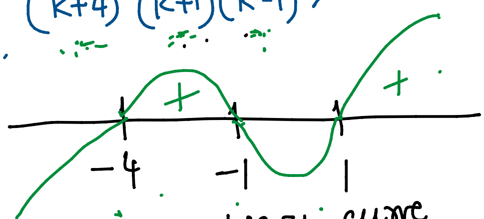
$abc > 0$

Case1: $a,b,c > 0$.

Case2: $a,b < 0$, $c > 0$

Case3: $a,c < 0$, $b > 0$.

Case4: $b,c < 0$, $a > 0$.



wavy curve method

$-4 < k < -1$

or $k > 1$

D. A sequence has $a_0 = 3$, and then for $n \geq 1$ the sequence satisfies $a_n = 8(a_{n-1})^4$. The

$r k > 1$

wavy curve method

D. A sequence has $a_0 = 3$, and then for $n \geqslant 1$ the sequence satisfies $a_n = 8\,(a_{n-1})^4$. The value of a_{10} is

(a) $\dfrac{2^{(2^{20})}}{3}$, (b) $\dfrac{6^{(2^{20})}}{3}$, (c) $\dfrac{3^{(2^{20})}}{2}$, (d) $\dfrac{18^{(2^{20})}}{2}$, (e) $\dfrac{6^{(2^{20})}}{2}$.

recursive relationship

$4^{10} = (2^2)^{10} = 2^{20}$.

4, 16, 64, 256

GP

$4^1, 4^2, 4^3, 4^4 \ldots 4^{10}$

for $a_{10} \to 3$

$a_0 = 3.$ $\quad a_1 = 8 a_0^4 = 8 \times 3^4$

$a_2 = 8 \times a_1^4 = 8[8 \times 3^4]^4 = 8 \times 8^4 \times 3^{16} = 8^5 \times 3^{16}$

$a_3 = 8 a_2^4 = 8[8^5 \times 3^{16}]^4 = 8 \times 8^{20} \times 3^{64} = 8^{21} 3^{64}$

$a_4 = 8 a_3^4 = 8(8^{21} \times 3^{64})^4 = 8 \times 8^{84} \times 3^{256} = 8^{85} \times 3^{256}$

$t_4 \quad t_5: 85 + 4^4 \quad t_6: t_5 + 4^5$

$1 \quad 5 \quad 21 \quad 85$

$4 \quad 16 \quad 64$

$4^1 \quad 4^2 \quad 4^3 \quad 4^4 \quad 4^5$

$a_{10} \to a_9 + 4^9.$

$$S = \frac{1 + 4^1 + 4^2 + 4^3 + \cdots + 4^9}{} = \frac{4^{10} - 1}{4 - 1} = \frac{4^{10} - 1}{3}$$

$(8)^{\frac{2^{20}-1}{3}}$

$$a_{10} = 8^{\frac{2^{20}-1}{3}} \times 3^{2^{20}}$$

$$= (2^3)^{\frac{(2^{20}-1)}{3}} \times 3^{2^{20}} = 2^{(2^{20}-1)} \cdot 3^{2^{20}} = \frac{2^{2^{20}} \times 3^{2^{20}}}{2^1} = \frac{6^{2^{20}}}{2}$$

$8 = 2^3$

E. If the expression $\left(x + 1 + \dfrac{1}{x}\right)^4$ is fully expanded term-by-term and like terms are collected together, there is one term which is independent of x. The value of this term is

(a) 10, (b) 14, (c) 19, (d) 51, (e) 81.

$\left[\left[\left(x + \frac{1}{x}\right) + 1\right]^2\right]^2.$

$\left[\left(x + \frac{1}{x}\right) + 1\right]^2 = \left(x + \frac{1}{x}\right)^2 + 1^2 + 2\left(x + \frac{1}{x}\right)$

$= x^2 + \frac{1}{x^2} + 2 + 1 + 2x + \frac{2}{x}$

$= x^2 + \frac{1}{x^2} + 3 + 2x + \frac{2}{x}$

$\left(x^2 + \frac{1}{x^2} + 2x + \frac{2}{x} + 3\right)$

$\left(x^2 + \frac{1}{x^2} + 2x + \frac{2}{x} + 3\right)$

$x^4 + 1 + 2x^3 + 2x + 3x^2$

$+1 + 2 + 2 + 3$

$x^4 + 1 + 2x^3 + 2x + 3x^2$ $\quad + \frac{1}{x^4} + \frac{2}{x} + \frac{2}{x^3} + \frac{3}{x^2}$

$+1$

$+4$ $\quad 2x^3 + 6x + 4x^2$ $\quad + \frac{2}{x}$

19 $\quad 4 \quad 2x$ $\quad + \frac{6}{x} + \frac{2}{x^3} + \frac{4}{x^2}$

$9 \quad 6x \quad 3x^2$ $\quad \frac{6}{x} \quad \frac{3}{x^2}$

F. Given that

$$\sin(5\theta) = 5\sin\theta - 20(\sin\theta)^3 + 16(\sin\theta)^5$$

for all real θ, it follows that the value of $\sin(72°)$ is

(a) $\sqrt{\frac{5+\sqrt{5}}{8}}$, (b) 0, (c) $-\sqrt{\frac{5+\sqrt{5}}{8}}$,

(d) $\sqrt{\frac{5-\sqrt{5}}{8}}$, (e) $-\sqrt{\frac{5-\sqrt{5}}{8}}$.

G. For all real n, it is the case that $n^4+1=(n^2+\sqrt{2}n+1)(n^2-\sqrt{2}n+1)$. From this we may deduce that n^4+4 is

(a) never a prime number for any positive whole number n.

(b) a prime number for exactly one positive whole number n.

(c) a prime number for exactly two positive whole numbers n.

(d) a prime number for exactly three positive whole numbers n.

(e) a prime number for exactly four positive whole numbers n.

H. How many real solutions x are there to the following equation?

$$\log_2(2x^3+7x^2+2x+3)=3\log_2(x+1)+1$$

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.