Q. Let $y(x)$ be a solution to the differential eqn $\dfrac{d^2y}{dx^2} - y = 0$ s.t $y(0) = 2$ and $y'(0) = 2\alpha$. Find all values of $\alpha \in [0, 1)$ s.t infimum of the set $\{y(x) \mid x \in \mathbb{R}\}$ is ≥ 1.

↳ $\min y(x) \geq 1$.

$$\frac{d^2y}{dx^2} - y = 0.$$

Trial soln: $y = e^{mx}$

AE: $m^2 - 1 = 0 \Rightarrow m = \pm 1$ [Real & distinct].

soln: $\boxed{y(x) = c_1 e^{x} + c_2 e^{-x}}$

$y(0) = 2 \Rightarrow \boxed{2 = c_1 + c_2}$ --- (i)

$y'(0) = 2\alpha \Rightarrow y'(x) = c_1 e^{x} - c_2 e^{-x}$

$\boxed{2\alpha = c_1 - c_2}$ ---- (ii)

Solving: $c_1 = (1+\alpha)$, $c_2 = (1-\alpha)$

Final soln: $y(x) = (1+\alpha) e^{x} + (1-\alpha) e^{-x}$; ∵ depends on α.

Given $\min y(x) \geq 1 \Rightarrow \boxed{y(x) \geq 1}$ ∴ [Find α s.t $y(x) \geq 1$]

$\Rightarrow (1+\alpha) e^{x} + (1-\alpha) e^{-x} \geq 1$.

$\Rightarrow (1+\alpha) e^{2x} + (1-\alpha) \geq e^{x}$.

$\Rightarrow \underbrace{(1+\alpha) e^{2x} - e^{x} + (1-\alpha)}_{\text{Quad in } e^{x}} \geq 0$.

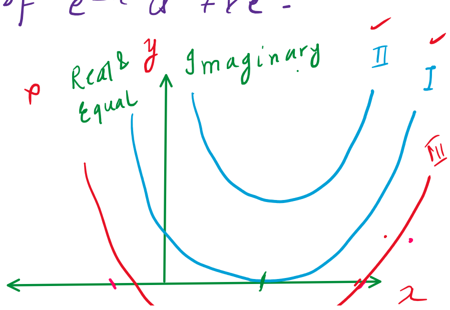
[Put $e^{x} = p \Rightarrow (1+\alpha) p^2 - p + (1-\alpha) \geq 0$]

As $\alpha \in [0, 1) \Rightarrow (1+\alpha) > 0 \Rightarrow$ coeff of e^{2x} is +ve.

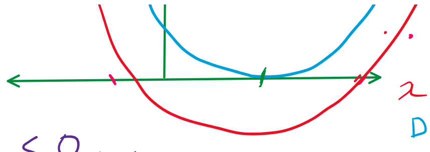
Note: $f(x) = ax^2 + bx + c$; $a > 0$.

Find cond s.t $\boxed{f(x) \geq 0 \ \forall x}$

Either real & equal Roots /



Either real & equal Roots / Imaginary roots $\Rightarrow D = b^2 - 4ac \le 0$



$$f(x) \ge 0 \text{ if } D \le 0 \quad \ldots\ldots (*)$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \underbrace{(1+\alpha)e^{2x} - e^{x} + (1-\alpha)}\ge 0 \;;\; (1+\alpha) > 0.$$

[This can be guaranteed on when discriminant $D = b^2 - 4ac \le 0$ for the quadratic].

$$\Rightarrow (-1)^2 - 4(1+\alpha)(1-\alpha) \le 0$$

$$\Rightarrow 1 - 4(1-\alpha^2) \le 0.$$

$$\Rightarrow 1 - 4 + 4\alpha^2 \le 0 \Rightarrow 4\alpha^2 \le 3$$

$$\Rightarrow \alpha^2 \le \frac{3}{4}$$

$$\Rightarrow -\frac{\sqrt{3}}{2} \le \alpha \le \frac{\sqrt{3}}{2}$$

As $\alpha \in [0, 1) \Rightarrow$ Reqd Range of $\alpha \in \left[0, \frac{\sqrt{3}}{2}\right]$.

Q. Consider the diff eqn $y' = \sqrt{y}$, $y(0) = \alpha$, $\alpha \ge 0$. Then the eqn has:

(a) atleast 2 solns if $\alpha = 0$ (b) no soln if $\alpha > 0$.

~~(c)~~ atleast 1 soln if $\alpha > 0$ ~~(d)~~ unique soln if $\alpha = 0$

$$\frac{dy}{dx} = \sqrt{y} \Rightarrow \text{Soln}: \quad 2\sqrt{y} = x + c$$

Given $y(0) = \alpha \Rightarrow 2\sqrt{\alpha} = c.$

$\therefore$ Particular soln : $2\sqrt{y} = x + 2\sqrt{\alpha}$.

$\alpha = 0 \Rightarrow 2\sqrt{y} = x.$

$$\sqrt{y} = \frac{x}{2} \Rightarrow y = \frac{x^2}{4} \quad \ldots \text{unique soln}$$

$\alpha > 0 \Rightarrow 2\sqrt{y} = x + 2\sqrt{\alpha}.$

$$\text{sq}: \quad 4 \cdot y = x^2 + 4\alpha + 2x\sqrt{\alpha}.$$

$$y = \frac{x^2 + 4\alpha + 2x\sqrt{\alpha}}{4} \quad \text{--- unique soln.}$$

$$y = \frac{x^2 + 4\alpha + 2x\sqrt{\alpha}}{4}$$ --- unique soln.

Q. One of the points which lies on the soln of the diff eqn $(y-x)\,dx + (x+y)\,dy = 0$, $y(0) = 1$ is:

(a) $(1, -2)$ (b) $(2, -1)$ (c) $(2, 1)$ (d) $(-1, 2)$.

$(y-x)\,dx + (x+y)\,dy = 0$. Form: $[\,M\,dx + N\,dy = 0\,]$.

Check for exactness:

$M = (y-x)$ $N = (x+y)$

For exactness: $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$ $\Rightarrow$ hence exact.

$$\Rightarrow y\,dx - x\,dx + x\,dy + y\,dy = 0.$$

$$(y\,dx + x\,dy) = x\,dx - y\,dy$$

$$\int d(xy) = \int x\,dx - \int y\,dy$$

$$\boxed{xy = \frac{x^2}{2} - \frac{y^2}{2} + c}$$ $\Rightarrow$ General soln:

$y(0) = 1$ $0 = 0 - \frac{1}{2} + c \Rightarrow c = \frac{1}{2}$

$$\therefore \boxed{xy = \frac{x^2}{2} - \frac{y^2}{2} + \frac{1}{2}}$$ --- $(2, 1)$ satisfies.

(*) Note: Suppose we have an exact differential eqn:

$M\,dx + N\,dy = 0$; where $M = M(x,y)$, $N = N(x,y)$; $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

General soln: $\boxed{\int_{y = \text{constant}} M\,dx + \int N\,(\text{excluding terms containing } x)\,dy = C}$

Given: $\underbrace{(y-x)}_{=M}\,dx + \underbrace{(x+y)}_{=N}\,dy = 0 \Rightarrow$ Exact.

Soln: $\int_{(y = \text{constant})} (y-x)\,dx + \int y\,dy = c$

Soln: $\int (y-x)\,dx + \int y\,dy = c$
(y = constant)

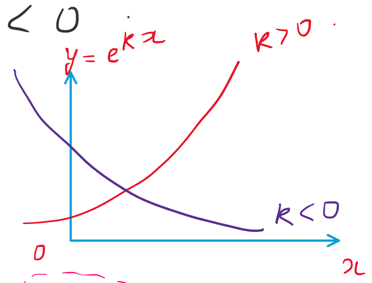
$$xy - \frac{x^2}{2} + \frac{y^2}{2} = c$$

8. The general soln of the diff eqn: $\frac{d^2y}{dx^2} + b\cdot\frac{dy}{dx} + cy = 0$ approaches zero as $x \to \infty$ if:

(a) $b < 0$, $c > 0$ (b) $b > 0$, $c < 0$.

(c) $b > 0$, $c > 0$ (d) $b < 0$, $c < 0$.

$$\frac{d^2y}{dx^2} + b\cdot\frac{dy}{dx} + cy = 0$$



Let $y = e^{mx}$ be trial soln.

A E: $m^2 + b\cdot m + c = 0 \Rightarrow$ Roots are m_1, m_2.

$$m_1, m_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

If $k>0$, $e^{kx} \to \infty$, $x \to \infty$

If $k<0$, $e^{kx} \to 0$, $x \to \infty$

Soln: $y = \underbrace{c_1 e^{m_1 x}} + \underbrace{c_2 e^{m_2 x}}$

as $x \to \infty$, $y \to 0$, possible when $m_1 < 0$, $m_2 < 0$.

Relation b/w b, c & Roots is AE:

Eqn: $m^2 + bm + c = 0$, Roots $= m_1, m_2$

$\Rightarrow$ $m_1 + m_2 = -b < 0 \Rightarrow b > 0$

$m_1 \cdot m_2 = c > 0 \Rightarrow c > 0$