Q. Let $X_1, X_2, \dots, X_n$ be a r.s from $Ber(p)$. Find the MLE of p.

$X \sim Ber(p)$, pmf: $f(x) = p^{x}(1-p)^{1-x}$, $x = 0, 1$.

(i) Likelihood fn: $L(p) = \prod_{i=1}^{n} f_p(x_i)$

$$= \prod_{i=1}^{n} p^{x_i}(1-p)^{(1-x_i)}$$

$$= p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$$

(ii) Log-likelihood fn: $\ell(p) = \ln L(p) = (\Sigma x_i)\ln p + (n - \Sigma x_i)\ln(1-p)$.

(iii) Max $\ell(p)$ w.r.t 'p' :-

$$\frac{\partial \ell(p)}{\partial p} = 0 \Rightarrow \frac{(\Sigma x_i)}{p} + \frac{(n-\Sigma x_i)}{(1-p)}(-1) = 0.$$

$$\Rightarrow \frac{(\Sigma x_i)}{p} = \frac{(n-\Sigma x_i)}{(1-p)}$$

$$\Rightarrow (1-p)\Sigma x_i = p(n - \Sigma x_i)$$

$$\Rightarrow \Sigma x_i - \cancel{p\Sigma x_i} = p \cdot n - \cancel{p\Sigma x_i}$$

$$\Rightarrow \Sigma x_i = p \cdot n \Rightarrow \boxed{\hat{p}_{MLE} = \frac{\Sigma x_i}{n} = \bar{x}}$$

Q. Let $X_1, X_2, \dots, X_n$ be a r.s from $f(x, \theta) = \frac{1}{2} e^{-|x-\theta|}$. Find the MLE of θ. Show that it is not unique.

(i) $L(\theta) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i - \theta|} = \left(\frac{1}{2}\right)^n \cdot e^{-\Sigma|x_i - \theta|}$.

(ii) $\ell(\theta) = \ln L(\theta) = -n \log 2 - \Sigma|x_i - \theta|$.

(iii) Choose 'θ' to max $\ell(\theta)$:-

$$\frac{\partial \ell(\theta)}{\partial \theta} = 0 \Rightarrow \text{Max}: \ -\sum |x_i - \theta|$$

$$\Rightarrow \text{Min}: \ \sum |x_i - \theta| \Rightarrow \text{This is minized when } \hat{\theta}_{MLE} = x_{Med}$$

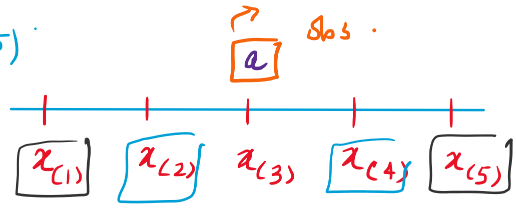
Note: $\sum |x_i - a|$ is minimized when $a = x_{Med}$.

$\therefore$ We know x_{Med} is not unique (depends on the size of the sample). Hence $\hat{\theta}_{MLE}$ is also not unique.

Note: $\sum_{i=1}^{n} |x_i - \overset{\text{Mean/ Med/ Mode}}{a}|$ is minimized when measured about median

Eg: $n=5$ s.t $x_{(1)} \le x_{(2)} \le x_{(3)} \le x_{(4)} \le x_{(5)}$.

Middle most obs.



$$= |x_1 - a| + |x_2 - a| + |x_3 - a| + |x_4 - a| + |x_5 - a|$$

$$= \{ |x_1 - a| + |x_5 - a| \} + \{ |x_2 - a| + |x_4 - a| \} + |x_3 - a|.$$

↳ choose 'a' s.t $x_1 \le a \le x_5$.

↳ choose 'a' s.t $x_2 \le a \le x_4$

choose a when $x_3 = a$.

Q. Let $X_1, X_2, \ldots, X_n$ be a r.s from $U(0, \theta)$. Find MLE of θ.

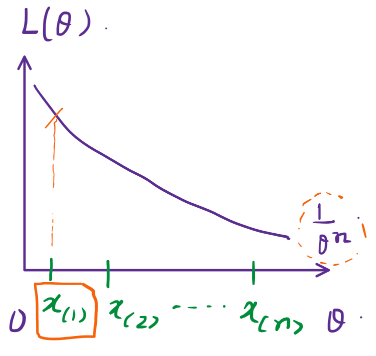
r.v $X \sim U(0,\theta)$: pdf: $f(x) = \frac{1}{\theta}, \ 0 \le x \le \theta$.

(i) $L(\theta) = \frac{1}{\theta^n}$

Obj: Choose θ to max $L(\theta)$ given the r.s.

(ii) $\ell(\theta) = -n \log \theta$.

(iii) $\frac{\partial \ell(\theta)}{\partial \theta} = 0 \Rightarrow \hat{\theta}_{MLE} = \} \times$.



We have: r.s $X_1, X_2, \ldots, X_n$.

Given: $U(0,\theta) \Rightarrow 0 \le x \le \theta$.

$\therefore \quad 0 \leq X_1, X_2, \ldots, X_n \leq \theta .$

Construct the ordered sample: $0 \leq x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)} \leq \theta$

Given the r.s, to max the likelihood fn $= \frac{1}{\theta^n}$, we need to choose the smallest possible value of θ from the r.s.

Given: $0 \leq x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)} \leq \theta .$

$\therefore$ smallest possible value of $\theta = x_{(1)}$.

$$\therefore \boxed{\hat{\theta}_{MLE} = x_{(1)}}$$

Q. Let $X_1, X_2, \ldots, X_n$ be a r.s from $U(\theta_1, \theta_2)$. Find the MLE of θ_1, θ_2.

$X \sim U(\theta_1, \theta_2)$ p.d.f $f(x) = \frac{1}{(\theta_2 - \theta_1)}, \quad \theta_1 \leq x \leq \theta_2 .$

(i) $L(\theta_1, \theta_2) = \prod_{i=1}^{n} f_{\theta_1, \theta_2}(x_i) = \prod_{i=1}^{n} \frac{1}{(\theta_2 - \theta_1)} = \frac{1}{(\theta_2 - \theta_1)^n}$ HW.

$\hat{\theta}_{1, MLE} =$ $\qquad \hat{\theta}_{2, MLE} =$