Q. $\sum u_n = \left(\frac{1}{2}\right)^p + \left(\frac{1\cdot 3}{2\cdot 4}\right)^p + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^p + \cdots\cdots - \quad \sum \frac{1}{n^p}$.

$$u_n = \left\{\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right\}^p$$

$$u_{n+1} = \left\{\frac{1\cdot 3\cdots(2n-1)(2n+1)}{2\cdot 4\cdots(2n)(2n+2)}\right\}^p$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p$$

- Comparision Test ✗
- Cauchy Root Test $(u_n)^{\frac{1}{n}}$

D-A. Ratio Test.
Raabe's Test.
Logarithmic Test
Gauss Test
$\left.\right\} \frac{u_n}{u_{n+1}}$

D-A Ratio Test: $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$.

Raabe's Test: $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\left(\frac{2n+2}{2n+1}\right)^p - 1\right]$

$$= \lim_{n\to\infty} n\left[\frac{(2n+2)^p - (2n+1)^p}{(2n+1)^p}\right]$$

Logarithmic Test: $\lim_{n\to\infty} n\cdot\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\, p\cdot\log\left(\frac{2n+2}{2n+1}\right)$

$$= p\cdot\lim_{n\to\infty} n\cdot\left[\log(2n+2) - \log(2n+1)\right]$$

$$= p\lim_{n\to\infty} n\left[\log\left(1+\frac{1}{2n+1}\right)\right].$$

$n \geq 1 \Rightarrow \frac{1}{2n+1} < 1 \Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\cdots$

$$= p\cdot\lim_{n\to\infty} n\left[\frac{1}{\underset{n}{2n+1}} - \frac{1}{2\underset{n}{(2n+1)^2}} + \frac{1}{3\underset{n}{(2n+1)^3}} + \cdots\right]$$

$$= p\cdot\lim\left[\frac{1}{n+1} - \frac{n}{2(2n+1)^2} + \frac{n}{3(2n+1)^3} + \cdots\right]$$

$$= p \cdot \lim_{n\to\infty} \left[\frac{1}{2+\frac{1}{n}} - \underbrace{\frac{n}{2(2n+1)^2}}_{\to 0} + \underbrace{\frac{n}{3(2n+1)^3}}_{\to 0} + \cdots \right]$$

$$= p/2 .$$

$p > 2 \Rightarrow$ convergent

$p < 2 \Rightarrow$ divergent

$p = 2 \Rightarrow$ Test fails.

$|-k| < 1$.

GP. $a = 1$, $r = -k$. $\quad \dfrac{1}{1+k}$.

Q. For the series: $1 - k + k^2 - k^3 + \cdots$, which of the following is FALSE:

(a) Divergent if $k = 1/3$. (c) Oscillatory if $k = -1$

(b) Divergent if $k = -3$ (d) Divergent if $k = -2$.

If $k = -1 \Rightarrow \sum u_n = 1 + 1 + 1 + 1 \cdots \Rightarrow$ Divergent.

Q. Check the convergence of series: $\dfrac{\log 2}{2^2} - \dfrac{\log 3}{3^2} + \dfrac{\log 4}{4^2} \cdots$

$u_n = (-1)^n \dfrac{\log(n+1)}{(n+1)^2}$ .... Alternating series.

Use Leibnitz Test: Check $\{u_n\}$ is monotonically decreasing and decreases to zero.

$$\lim_{n\to\infty} \frac{\log(n+1)}{(n+1)^2} = 0 .$$

$$u_n = \frac{\log(n+1)}{(n+1)^2} \qquad u_{n+1} = \frac{\log(n+2)}{(n+2)^2}$$

Check: $u_{n+1} - u_n < 0$.

$f(x) = \frac{\log x}{x^2} \Rightarrow$ show $f' < 0 \Rightarrow$ monotonically decreasing

$$f' = \frac{x^2 \cdot \left(\frac{1}{x}\right) - \log x \cdot 2x}{x^4} = \frac{x - 2x\log x}{x^4} = \frac{1 - 2\log x}{x^3}.$$

Q. If p and q are positive real numbers, then the series:

$$\sum u_n = \frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^p} + \cdots\cdots$$

is convergent if

(a) $p < q - 1$ ✓   (c) $p \geq q - 1$

(b) $p < q + 1$ (d) $p \geq q + 1$

$$u_n = \frac{(n+1)^p}{n^q}$$

$$u_{n+1} = \frac{(n+2)^p}{(n+1)^q}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^p}{n^q} \cdot \frac{(n+1)^q}{(n+2)^p} = \left(\frac{n+1}{n+2}\right)^p \left(1 + \frac{1}{n}\right)^q.$$

Logarithmic Test: $\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right)$

$$= \lim_{n\to\infty} n \cdot \left[p \log\left(\frac{n+1}{n+2}\right) + q \log\left(1 + \frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} n \left[p \log\left(1 - \frac{1}{n+2}\right) + q \log\left(1 + \frac{1}{n}\right)\right]$$

$\log(1-x)$, $\frac{1}{n+2} < 1$ $\log(1+x)$ $\frac{1}{n} < 1$.

$$= -p + q$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots\cdots$$

$\log\left(1 - \frac{1}{\ }\right) = -\frac{1}{\ } - \frac{1}{\ } - \cdots\cdots$

$$\log\left(1-\frac{1}{n+2}\right) = -\frac{1}{n+2} - \frac{1}{2(n+2)^2} - \cdots$$

$$\log\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots$$

$-p+q > 1 \Rightarrow$ For convergence

$p < q-1$ (a)

Q. $\sum u_n = \frac{2}{1} + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \cdots$

$$u_n = \frac{n+1}{n^2} = \underset{\text{Div}}{\underset{\Downarrow}{\frac{1}{n}}} + \underset{\text{Conve}}{\underset{\Downarrow}{\frac{1}{n^2}}}$$

$\sum u_n$ will be divergent.

$$v_n = \frac{n}{n^2} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n+1}{n^2}\cdot n = \lim_{n\to\infty} 1+\frac{1}{n} = 1$$

$\therefore$ As $\sum v_n$ diverges, $\sum u_n$ also diverges.

$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$

$\left\{\frac{1}{n}\right\} \to 0$ [Convergent sequence]

$\sum\left(\frac{1}{n}\right) \Rightarrow p$-series $p=1$ [Divergent series].

$\sum\left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

HW

Q. $\sum u_n = \sum \left(\frac{n - 2\log n}{2n}\right)^n$

Q. $\sum u_n = \sum \sqrt{\frac{2^n - 1}{3^n - 1}}$