Q. Let $Ax = b$ be a system of equations where $A_{m \times n}$. Consider the set $P = \{ \underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n, b \}$, where $\underset{\sim}{a}_i$ are columns of matrix A.
Set P is :

(i) always l.i

(ii) always l.d. ✓

(iii) l.d iff $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ are l.d

(iv) l.d iff $m = n$.

System $A_{m \times n} \, x_{n \times 1} = b_{m \times 1}$. $\quad (\underset{\sim}{a}_i)_{m \times 1}$

$P = \{ (\underset{\sim}{a}_1)_{m \times 1}, \ldots, (\underset{\sim}{a}_n)_{m \times 1}, b_{m \times 1} \}$. No. of vectors $= (n+1)$.

$n+1 > m \Rightarrow$ l.d.

$Ax = b$.

$$\begin{bmatrix} \underset{\sim}{a}_1 & \underset{\sim}{a}_2 & \cdots & \underset{\sim}{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \to b.$$

$\underset{\sim}{a}_1 \quad \underset{\sim}{a}_2 \quad \underset{\sim}{a}_n$

$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \ldots$ [Linear combination of $a_{11}, a_{12}, \cdots a_{1n}$ $\;\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$]

$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$

$- - - - - - - - - - - - -$

$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$.

$\underset{\sim}{b}$ can be expressed as a linear combination of $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$

$P = \{ \underbrace{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n}, \underset{\sim}{b} \} \Rightarrow$ l.d.

$P = \{ \underbrace{\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n}, \underset{\sim}{b} \} \Rightarrow$ L.d.

$$x_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2 + \cdots + x_n \underset{\sim}{a}_n = \underset{\sim}{b}.$$

$(1, 2, \cdots, n) \to (1^2, 2^2, \cdots, n^2)$

$f: D \to R$

$f(x) = x^2$

Linear Transformations:-

Consider 2 vector spaces U and V. Consider a transformation $T: U \to V$. T is said be linear iff:-

(i) $T(\underset{\sim}{\alpha} + \underset{\sim}{\beta}) = T(\underset{\sim}{\alpha}) + T(\underset{\sim}{\beta})$, $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in U$

(ii) $T(c\underset{\sim}{\alpha}) = c \cdot T(\underset{\sim}{\alpha})$, c is a scalar, $\underset{\sim}{\alpha} \in U$

Merging: $T(c_1 \underset{\sim}{\alpha} + c_2 \underset{\sim}{\beta}) = c_1 T(\underset{\sim}{\alpha}) + c_2 T(\underset{\sim}{\beta})$,

where c_1, c_2 are scalar, $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in U$

Q. Check for linear transformation:

i) $f: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad f(x, y) = (x + 2y, x - y, -2x + 3y)$

ii) $g: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad g(x, y) = (x + 1, y + 2)$

i) $f: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad f(x, y) = (x + 2y, x - y, -2x + 3y)$

Defn: $T(c_1 \underset{\sim}{\alpha} + c_2 \underset{\sim}{\beta}) = c_1 T(\underset{\sim}{\alpha}) + c_2 T(\underset{\sim}{\beta})$, $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in U$, c_1, c_2 are scalars

Consider $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2)$ $\quad \underset{\sim}{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$

and 2 scalars c_1, c_2.

LHS: $T(c_1 \underset{\sim}{\alpha} + c_2 \underset{\sim}{\beta}) = T(c_1(\alpha_1, \alpha_2) + c_2(\beta_1, \beta_2))$

$$= T(\underbrace{c_1\alpha_1 + c_2\beta_1}_{x}, \underbrace{c_1\alpha_2 + c_2\beta_2}_{y})$$

$$= (\underbrace{c_1\alpha_1 + c_2\beta_1 + 2(c_1\alpha_2 + c_2\beta_2)}, \underbrace{c_1\alpha_1 + c_2\beta_1 - (c_1\alpha_2 + c_2\beta_2)},$$

$$\underbrace{-2(c_1\alpha_1 + c_2\beta_1) + 3(c_1\alpha_2 + c_2\beta_2)}\ )$$

RHS: $c_1 T(\underset{\sim}{\alpha}) + c_2 . T(\underset{\sim}{\beta}) = c_1 T(\alpha_1, \alpha_2) + c_2 T(\beta_1, \beta_2)$

$$= c_1 (\alpha_1 + 2\alpha_2, \alpha_1 - \alpha_2, -2\alpha_1 + 3\alpha_2) + c_2(\beta_1 + 2\beta_2, \beta_1 - \beta_2, -2\beta_1 + 3\beta_2).$$

$$= (\underbrace{c_1\alpha_1 + 2c_1\alpha_2 + c_2\beta_1 + 2c_2\beta_2}, \underbrace{c_1\alpha_1 - c_1\alpha_2 + c_2\beta_1 - c_2\beta_2},$$

$$\underbrace{-2c_1\alpha_1 + 3c_1\alpha_2 - 2c_2\beta_1 + 3c_2\beta_2})$$

f is a linear transformation.

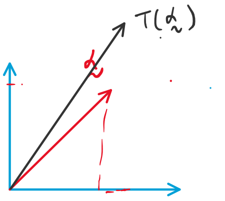
(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad g(x, y) = (x+1, y+2)$

Consider $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$

LHS, $T(c_1\underset{\sim}{\alpha} + c_2\underset{\sim}{\beta}) = T[c_1(\alpha_1, \alpha_2) + c_2(\beta_1, \beta_2)]$

$$= T[(\overbrace{c_1\alpha_1 + c_2\beta_1}^{x}), (\overbrace{c_1\alpha_2 + c_2\beta_2}^{y})]$$

$$= (c_1\alpha_1 + c_2\beta_1 + 1, c_1\alpha_2 + c_2\beta_2 + 2)$$



$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

$T(\alpha) = \begin{bmatrix} \alpha_1 + 1 \\ \alpha_2 + 2 \end{bmatrix}$

RHS, $c_1 T(\underset{\sim}{\alpha}) + c_2 T(\underset{\sim}{\beta}) = c_1 T(\alpha_1, \alpha_2) + c_2 T(\beta_1, \beta_2)$

$$= c_1(\alpha_1 + 1, \alpha_2 + 2) + c_2(\beta_1 + 1, \beta_2 + 2)$$

$$= (c_1\alpha_1 + c_1 + c_2\beta_1 + c_2, c_1\alpha_2 + 2c_1 + c_2\beta_2 + 2c_2)$$

g is not a linear transformation.

Null space of a Linear Transformation:

Consider a linear transformation $T : U \to V$, then

$$N(T) = \{ \underset{\sim}{\alpha} \in U \text{ s.t } T(\underset{\sim}{\alpha}) = \underset{\sim}{0} \}$$

Nullity: The dimension of $N(T)$ is called its nullity.

Range of a Linear Transformation:-

Consider a linear transformation $T: U \to V$, then

$$R(T) = \{ \underset{\sim}{v} \in V \text{ s.t } T(\underset{\sim}{u}) = \underset{\sim}{v} , \underset{\sim}{u} \in U \}.$$

Rank of T: Dimension of $R(T)$

(*) Rank-Nullity Theorem:

Rank of T + Nullity of T = Dimension of U.

Q. Show that the 3 vectors $\alpha_1 = (0, 2, -4)$ $\alpha_2 = (1, -2, -1)$ $\alpha_3 = (1, -4, 3)$ is l.d.

$$\Delta = \begin{vmatrix} 0 & 2 & -4 \\ 1 & -2 & -1 \\ 1 & -4 & 3 \end{vmatrix} \neq 0 \quad \text{l.i.}$$

$$c_1(0, 2, -4) + c_2(1, -2, -1) + c_3(1, -4, 3) = (0, 0, 0)$$

Q. Show that $(1, 1, 2, 4)$ $(2, -1, -5, 2)$ $(1, -1, -4, 0)$ $(2, 1, 1, 6)$

$$\Delta = \begin{vmatrix} \cancel{1} & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{vmatrix} = 1 \begin{vmatrix} -1 & -5 & 2 \\ -1 & -4 & 0 \\ 1 & 1 & 6 \end{vmatrix} - 1 \begin{vmatrix} 2 & -5 & 2 \\ 1 & -4 & 0 \\ 2 & 1 & 6 \end{vmatrix} +$$

$$\underset{3\times 3}{\Delta} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \xrightarrow[R_1]{\text{Expand}} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} & \\ & \end{vmatrix} + a_{13} \begin{vmatrix} & \\ & \end{vmatrix}$$

For 3×3: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$.

$(-1)^{i+j}$ $\Rightarrow$ To determine the sign of the minor(i,j)

$\begin{vmatrix} + & - & + & - \\ - & + & - & + \end{vmatrix}$ 4×4.

sign of the minor(i,j) $\begin{vmatrix} - & + & - & + \end{vmatrix}$ 4×4.