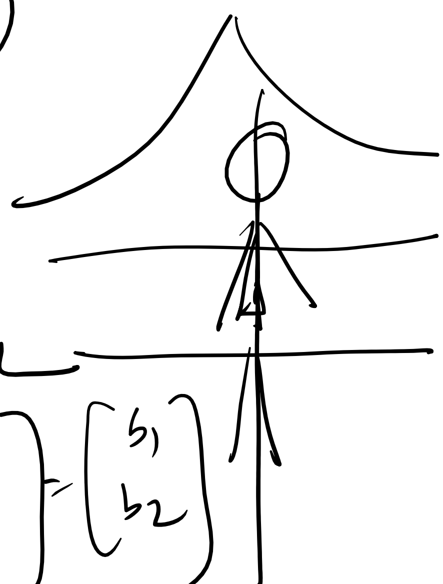
MATHEMATICS

90623-95723

Session on eigen values
equation system
Advanced functions of linear algebra

Symmetric structure



$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\boxed{AX = B}$$

Cramer's Rule

$x_1 = \frac{\Delta x}{\Delta}$ $\quad y = \frac{\Delta y}{\Delta}$

$\Delta x = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$ $\quad \Delta y = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$

$x = \frac{\Delta x}{\Delta}$ $\quad y = \frac{\Delta y}{\Delta}$ $\rightarrow$ Consistent & unique soln

$\Delta = 0$ atleast $\Delta x, \Delta y \neq 0$ Inconsistent...

$\Delta = 0$ $\quad \Delta x = \Delta y = 0$ $\quad \infty$ soln

(II) $Ax=0$

$|A|\neq 0$ only trivial sol^n

$|A|=0$ ∞ many sol^n

$|A|=0$ then $(\text{adj} A)B=0$ or $B=0$

\# $B=0$ $Ax=0$ homo eqn.

TIFR Always Consistent

$\boxed{Ax=0}$ has unique sol^n if Rank$(A)=n$

∞ no of sol^n if $R(A)<n$

Non-homogeneous

$Ax=b$ m equations $m\neq n$
n variables

$$X=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}_{n\times1} \qquad b=\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}_{m\times1}$$

$Ax=b$ has a sol^n
iff $b\in$ Column space of A
$R(A)=R[A:b]$

$$0=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}_{n\times1}$$

$Ax=0$ has non-trivial sol^n
iff
$\text{Ker}(A)\neq\{0\}$
nullity$(A)\geq 1$
$R(A)<n$
A is not one-one

$R(A) \neq \mathbb{R}^2$

A is not one-one

Characteristic eq & Eigen value

$$\begin{bmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)^2 = 0 \qquad \boxed{\lambda = 3, 3}$$

Geometric multiply: λ

dimension of eigen space

(set of linearly independent eigen vectors

corresponding to eigenvalue λ)

$\boxed{\lambda = 3}$

$$(\lambda I - A)X = 0$$

$$(3I - A)X = 0$$

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2y = 0 \qquad \boxed{y = 0}$$

X is free variable

$x = 1$ $(1, 0)$ lin ind ~~vect~~ eigen vector of A for $\lambda = 3$

Basis of Eigen Space $= \{(1,0)\}$

Geomet multi λ is 1

Eigen value

A Inverse $A^{-1} \rightarrow \left(\frac{1}{\lambda}\right)$

(A) Inverse | A^{-1} $\to$ $\frac{1}{\lambda}$

Shift | $A + cI$ $(\lambda + c)$

Similar | $B = P^{-1}AP$ $\lambda(B) = \lambda(A)$

Unitary | $AA^\theta = I = A^\theta A$, $A^\theta = A^{-1}$ All λ, $|\lambda| = 1$

Polynomial | $P(A)$ $P(\lambda)$

Involution | $A^2 = I$ $\lambda = \pm 1$

* Idempotent | $A^2 = A$ $\lambda = 0, 1$

+ve definite matrix | $x^T A x > 0$ $\boxed{\lambda > 0}$

+ve semi def " | $x^T A x \geq 0$ $\lambda \geq 0$

-ve def | $x^T A x < 0$ $\lambda < 0$

-ve semi-def | $x^T A x \leq 0$ $\lambda \leq 0$

Nilpotent | $A^n = 0$ $n \in N$ $\boxed{\lambda = 0}$

Real orthogonal | $A^T = A^{-1}$ $|\lambda| = 1$

Companion Matrix

monic polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & & & & \\ 0 & \cdots & \cdots & 1 & -a_{n-1} \end{bmatrix}_{n \times n}$$

$$C_A(x) = \det(xI - A) = \det \begin{bmatrix} x & 0 & \cdots 0 & a_0 \\ -1 & x & \cdots 0 & a_1 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & -1 \cdots & x + a_{n-1} \end{bmatrix}$$

$$= (x^n + a_{n-1}x^{n-1} + \cdots + a_0)(-1)^{n-1}(-1)^{n+1}$$

$= (x^n + a_{n-1}x^{n-1} + \dots + a_0)(-1)^{n+1}(-1)^{n+1}$

$$\begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Diagonalization of Matrix

Similar to a diagonal matrix

$\exists\, D = \mathrm{diag}(d_1, d_2 \dots d_n)$
non-singular matrix P for $A = PDP^{-1}$

$$\boxed{P^{-1}AP = D}$$

Rules

(I) n linearly independent eigen vectors

(II) Geometric multiplicity of EV = Algebraic multiplicity

$A \to$ diag A^{-1} B also diag

$A \to$ diag A^T — —

$A \to$ diag $P(A)$ — —

$A \to$ diag $A + CI$ — —

A B $n \times n$ $\left.\begin{matrix} A+B \\ A-B \end{matrix}\right\}$ may not be diagonalizable

Diagonalizability of different matrices ..

Null ✓	non 0 nilpotent X
Rank 1 ✓	Rank 1 with TR = 0 X
Normal ✓	non 0 strict, upper triangular X

Rank ✓   Rank ...
normal ✓   non 0 Strict upper triangle ✗
              " " " lower " ✗
              AM ≠ GM of eigen value ✗

$(64)^{1/3}$ $= 4$ 8, 4, 2

$\frac{8+4+2}{3} = 14/3 = 4.6$

$\boxed{AM \geq GM}$

for what value a, b

$$\begin{bmatrix} 0 & a & 0 \\ b & 0 & b \\ 0 & a & 0 \end{bmatrix}$$

$$A - xI = \begin{bmatrix} -x & a & 0 \\ b & -x & b \\ 0 & a & -x \end{bmatrix}$$

$$|A - xI| = -x(x^2 - ab) - a(-xb - 0)$$
$$= (-x^3 + 2xab)$$

$(A - xI) = 0$
$x^3 - 2xab = 0$

$x(x^2 - 2ab) = 0$
$x = 0, \pm\sqrt{2ab}$

If $a \neq 0, b \neq 0$ all ~~...~~ E.v. are distinct
A is diagonalizable

If $a = 0, b \neq 0$ A → 0, 0, 0   E v

Algebraic multip = 3
Geometric multip → 3
nullity $(A - 0.I) = 3$ $R = 0$

GM ≠ AM   A is not diagonalizable

$a \neq 0 \quad b = 0$
$a = 0, b = 0$

A
$A = 0, 0, 0$
$Am = 3$
$Gm = 3$
A is [illegible]

Algebraic Multiplicity

no of times it appear as a root of CP
How many times a single EV is repeated as a sol'n

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 4\lambda + 4 = 0$
$\lambda = 2, 2 \quad AM = 2$

AM

dimension of the eigenspace

no of linearly independent E.V.

$\boxed{\lambda = 2}$

$(A - 2I)v = [0, 1]$
$v = [1, 0]$
$GM = 1$

$AM \neq GM$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/w & 1/w^2 \\ 1 & 1/w^2 & 1/w^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & -w & 0 \\ 1 & -w^2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w \\ 1 & w & w^2 \end{bmatrix} = \begin{bmatrix} 0 & 1-w^2 & 1-w \\ 1-w & 0 & 1-w^2 \\ 1-w^2 & 1-w & 0 \end{bmatrix}$$

$$\boxed{\lambda = 3}$$

$1+1-(w+w^2)$

$1+1+1 = 3$

$(3, -3, 0)$

4×4 Skew-Symmetric orthogonal real matrix

$\pm i$ $\quad \pm i, \pm 1 \quad \pm 1 \quad 0, \pm i$

$AA' = I \quad |A| = \pm 1$

all E.V → non zero

Skew Symmetric matrix — 0 — purely imaginary