10 February 2024 16:30

Trigonometric equations

Equation of the Form $a\cos\theta + b\sin\theta = c$

$\boxed{a = r\sin\alpha. \quad b = r\cos\alpha.} \Rightarrow r^2 = a^2+b^2, \; \theta = \tan^{-1}\left(\frac{a}{b}\right)$

$r\cos\alpha\sin\theta + r\sin\alpha\cos\theta = c$

$\cos\alpha = \frac{b}{r}, \; \sin\alpha = \frac{a}{r}.$

$r\sin(\theta+\alpha) = c.$

$\boxed{\sin(\theta+\alpha) = \frac{c}{r}} \Rightarrow$ soln is only possible if $\frac{|c|}{r} \le 1$

Solve $\sin x + \sqrt{3}\cos x = \sqrt{2}$.

$a = 1 \quad b = \sqrt{3} \quad c = \sqrt{2}.$

$\boxed{r\cos\theta = 1 \quad r\sin\theta = \sqrt{3}} \Rightarrow r^2 = a^2+b^2 = 4$

$\boxed{r = 2}$

$\theta = \tan^{-1}(\sqrt{3})$

$\boxed{\theta = \frac{\pi}{3}}$

$r\cos\theta\sin x + r\sin\theta\cos x = \sqrt{2}.$

$r\sin(\theta+x) = \sqrt{2}.$

$\sin(\theta+x) = \frac{\sqrt{2}}{r} = \frac{1}{\sqrt{2}}.$

$\sin(\theta+x) = \sin\frac{\pi}{4}.$

$\sin x = \sin\theta$

$\Rightarrow x = n\pi + (-1)^n\theta.$

$\theta + x = n\pi + (-1)^n\frac{\pi}{4}.$

$x = n\pi + (-1)^n\frac{\pi}{4} - \theta.$

$x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}.$

$\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$.

$1 + \sin x = \sqrt{3}\cos x. \Rightarrow \sqrt{3}\cos x - \sin x = 1.$

$a = \sqrt{3}, \; b = -1, \; c = 1$

$\boxed{r = \sqrt{a^2+b^2} = 2}$

$\frac{|c|}{r} = \frac{1}{2} < 1$

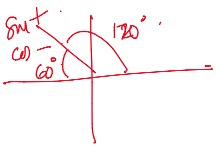
$\sqrt{3} = r\sin\theta, \; -1 = r\cos\theta.$

$r\sin\theta\cos x + r\cos\theta\sin x = 1$

$\tan\theta = \frac{\sqrt{3}}{-1}.$

$r\sin(x+\theta) = 1$

$\theta =$



$\sin(x+\theta) = \frac{1}{r} = \frac{1}{2} = \sin\frac{\pi}{6}.$

$x + \theta = n\pi + (-1)^n\frac{\pi}{6}.$

$x = n\pi + (-1)^n\frac{\pi}{6} - \theta = n\pi + (-1)^n\frac{\pi}{6} - \frac{2\pi}{3}.$

for $n=1$ $\quad x = \pi - \frac{\pi}{6} - \frac{2\pi}{3} = \frac{\pi}{6}.$

for $n=2$ $\quad x = 2\pi + \frac{\pi}{6} - \frac{2\pi}{3} = \frac{9\pi}{6} = \frac{3\pi}{2}.$

for $n=3$ $\quad x = 3\pi - \frac{\pi}{6} - \frac{2\pi}{3} = \frac{13\pi}{6}.$

Equation of the Form

$$a_0 \sin^n x + a_1 \sin^{n-1} x \cos x + a_2 \sin^{n-2} x \cos^2 x + \ldots + a_n \cos^n x = 0,$$

$\leftarrow (a \sin x + b \cos x)^n$

divide by $\cos^n x$. $\rightarrow$ convert the equation into a polynomial in $\tan x$.

$$a_0 \tan^n x + a_1 \tan^{n-1} x + a_2 \tan^{n-2} x + \ldots + a_n = 0.$$

Solve using polynomial rules.

$$5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4.$$

divide by $\cos^2 x$.

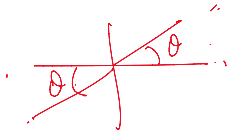
$5\tan^2 x - 7\tan x + 16 = 4\sec^2 x.$

$5\tan^2 x - 7\tan x + 16 = 4(\tan^2 x + 1)$

$\tan^2 x - 7\tan x + 12 = 0.$

$(\tan x - 3)(\tan x - 4) = 0.$

$\tan x = 3, 4.$ $\quad x = n\pi + \arctan(3),\ n\pi + \arctan(4)$



$\tan x = \tan\theta$

$x = n\pi + \theta.$

Equation of the Form

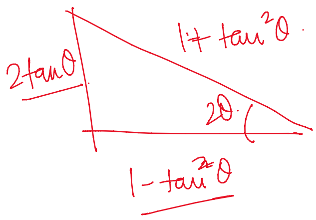
$$R(\sin kx, \cos nx, \tan mx, \cot lx) = 0,$$

use the multiple angle identities.

$\sin 2\theta$, $\tan 2\theta$

$\downarrow$ $\quad \downarrow$

$2\sin\theta\cos\theta$ $\quad \boxed{\frac{2\tan\theta}{1-\tan^2\theta}}$



$\sin 2\theta = 2\tan\theta$

$$\boxed{\dfrac{}{1-\tan^2\theta}}$$

$1-\tan^2\theta$

$\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

$\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$$(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$$

$\cos x = \dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}$ $\quad \sin x = \dfrac{2\tan x/2}{1+\tan^2 x/2}$

$\tan x = \dfrac{2\tan x/2}{1-\tan^2 x/2}$

$$\left(\frac{1-2\tan x/2 - \tan^2 x/2}{1+\tan^2 x/2}\right)\left(\frac{4\tan x/2}{1-\tan^2 x/2} + \frac{1+\tan^2 x/2}{1-\tan^2 x/2}\right) + 2 = 0$$

$$\boxed{\tan x/2 = t}$$

$$\left(\frac{1-2t-t^2}{1+t^2}\right)\left(\frac{4t+1+t^2}{1-t^2}\right) + 2 = 0$$

$$-t^4 - 2t^3 + \cancel{t^2} - 4t^3 - 8t^2 + 4t - \cancel{t^2} - 2t + 1 + 2(1-t^4) = 0$$

$$-3t^4 - 6t^3 - 8t^2 + 2t + 3 = 0$$

$$3t^4 + 6t^3 + 8t^2 - 2t - 3 = 0$$

	LHS
$t=0$	-3
$t=1$	12

1

$\frac{1}{2}, \frac{1}{3}, \frac{1}{\sqrt3}$

$\frac{1}{\sqrt3}$

$t = \frac{1}{\sqrt3} \to (\sqrt3 t - 1)$

$$(3t^4 + 8t^2 - 3) + (6t^3 - 2t) = 0$$

$$(3t^4 + 9t^2 - t^2 - 3) + 2t(3t^2-1) = 0$$

$$[3t^2(t^2+3) - 1(t^2+3)] + 2t(3t^2-1) = 0$$

$$(t^2+3)(3t^2-1) + 2t(3t^2-1) = 0$$

$$(3t^2-1)(t^2+3+2t) = 0$$

$$(3t^2-1)(t^2+2t+3) = 0$$

$$\boxed{t = \pm\frac{1}{\sqrt3}}$$

$\tan x/2 = \pm\frac{1}{\sqrt3} = \tan\frac{\pi}{6}, \frac{5\pi}{6}$

$x/2 = n\pi + \frac{\pi}{6},\ n\pi + \frac{5\pi}{6}$

$x = 2n\pi + \frac{\pi}{3},\ 2n\pi + \frac{5\pi}{3}$

Equation of the Form

$R(\sin x + \cos x, \sin x \cdot \cos x) = 0,$

$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x\cos x$

$(\sin x + \cos x)^2 = 1 + 2\sin x\cos x$

$\boxed{\sin x + \cos x = t}$ $\quad \boxed{\sin x\cos x = \dfrac{t^2-1}{2}}$

$\sin x+\cos x-2\sqrt{2}\sin x\cos x=0$

$\boxed{\sin x+\cos x=t}$ $\sin x\cos x=\frac{t^2-1}{2}.$

$t-\cancel{2}\sqrt{2}\left(\frac{t^2-1}{\cancel{2}}\right)=0.$

$t-\sqrt{2}t^2+\sqrt{2}=0.$

$\sqrt{2}t^2-t-\sqrt{2}=0.$ $\Rightarrow t=\frac{1\pm\sqrt{9}}{2\sqrt{2}}.$ $=\frac{1\pm 3}{2\sqrt{2}}$

$t=\frac{4}{2\sqrt{2}}, \frac{-2}{2\sqrt{2}}$ $\Rightarrow \boxed{t=\sqrt{2}, -\frac{1}{\sqrt{2}}.}$

Find the number of integral values of k for which equation $7\cos x+5\sin x=2k+1$ has at least one solution.

Solve $2\sin^2 x-5\sin x\cos x-8\cos^2 x=-2.$

Solve the equation $(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$.