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functions

Assume that $f(1) = 0$ and that for all integers m and n, $f(m + n) = f(m) + f(n) + 3(4mn - 1)$, then $f(19) =$

(A) 2049 (B) 2098 (C) 1944 (D) 1998

$m = 1 \quad n = 1$

$\Rightarrow f(2) = 2f(1) + 3(4 \times 1 \times 1 - 1) = 9$

$f(4) = 2f(2) + 3(4 \times 2 \times 2 - 1) = 18 + 3(15) = 63.$

$f(8) = 2f(4) + 3(4 \times 4 \times 4 - 1) = 2 \times 63 + 3 \times 63 = 5 \times 63 = 315.$

$\Rightarrow f(16) = 2f(8) + 3(4 \times 8 \times 8 - 1) = 2 \times 315 + 3(255) = 630 + 765 = 1395$

$m = 16, n = 2$ $\quad f(18) = f(16) + f(2) + 3(4 \times 16 \times 2 - 1) = 1395 + 9 + 3(127) = 1404 + 381 = 1785$

$m = 18, n = 1$ $\quad f(19) = f(18) + f(1) + 3(4 \times 18 \times 1 - 1) = 1785 + 3 \times 71 = 1785 + 213 = 1998$

2. $f(x) = \{x\} + \{x + 1\} + \{x + 2\}........\{x + 99\}$, then $[f(\sqrt{2})]$ where {.} denotes fractional part function & [.] denotes the greatest integer function =

$x+0.$ $\quad x+99$

(A) 5050 (B) 4950 (C) 41 (D) 14

$x = [x] + \{x\}$ $\quad x = 1.5 \quad [x] = 1 \quad \{x\} = 0.5$

$\boxed{\{x\} = x - [x]}$

$\frac{9900}{2} = 4950.$

$f(x) = x - [x] + x + 1 - [x+1] + x + 2 - [x+2] + \cdots + x + 99 - [x+99]$

$= 100x + \frac{99 \times 100}{2} - [[x] + [x+1] + [x+2] + \cdots + [x+99]]$

$= 100x + 4950 - [[x] + [x+1] + [x+2] + \cdots + [x+99]]$

$f(\sqrt{2}) = 100 \times \sqrt{2} + 4950 - [[\sqrt{2}] + [\sqrt{2}+1] + [\sqrt{2}+2] + \cdots + [\sqrt{2}+99]]$

$\sqrt{2} = 1.414 \quad \sqrt{2} + 1 = 2.414 \quad \sqrt{2} + 2 = 3.414 \quad \sqrt{2} + 99 = 100.414$

$f(\sqrt{2}) = 141.4 + 4950 - [1 + 2 + 3 + \cdots + 100]$

$= 141.4 + 4950 - \frac{100 \times 101}{2} = 141.4 + 4950 - 5050 = 141.4 - 100 = 41.4$

$[f(\sqrt{2})] = 41$

3. If $f_0(x) = x/(x + 1)$ and $f_{n+1} = f_0 \text{ o } f_n$ for n = 0, 1, 2,......, then $f_n(x)$ is -

(A) $f_n(x) = \frac{x}{(n+1)x+1}$ (B) $f_0(x)$ (C) $\frac{nx}{nx+1}$ (D) $\frac{x}{nx+1}$

$f_0 \circ f_n = f_0[f_n(x)].$ $\quad f \circ g = f[g(x)].$

$f_{n+1}(x) = f_0[f_n(x)] = f_n(x)$ $\quad \boxed{f_{n+1}(x) = f_n(x)}$ recursive relationship

$f_0 \circ f_n = f_0\,[f_n(x)]$.

$f_{n+1}(x) = f_0\,[f_n(x)] = \dfrac{f_n(x)}{f_n(x)+1}$ $\qquad \boxed{f_{n+1}(x) = \dfrac{f_n(x)}{f_n(x)+1}}$ recursive relationship

$n=0$. $\quad f_1(x) = \dfrac{f_0(x)}{f_0(x)+1} = \dfrac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \dfrac{x}{2x+1}$

$n=1$ $\quad f_2(x) = \dfrac{f_1(x)}{f_1(x)+1} = \dfrac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \dfrac{x}{3x+1}$

$n=2$ $\quad f_3(x) = \dfrac{f_2(x)}{f_2(x)+1} = \dfrac{\frac{x}{3x+1}}{\frac{x}{3x+1}+1} = \dfrac{x}{4x+1}$

$f_n(x) = \dfrac{x}{(n+1)x+1}$

6. If $f(x)$ be a function such that $f(x+1) = \dfrac{f(x)-1}{f(x)+1}$, $\forall\ x \in N$ and $f(1) = 2$ then $f(999)$ is -

(A) –3 (B) 2 (C) $\frac{1}{3}$ (D) $-\frac{1}{2}$

$f(a) = f\left[\text{Rem}\left(\frac{a}{4}\right)\right] = f(1)$

$x=1$. $\quad f(2) = \dfrac{f(1)-1}{f(1)+1} = \dfrac{2-1}{2+1} = \dfrac{1}{3}$.

$x=2$ $\quad f(3) = \dfrac{f(2)-1}{f(2)+1} = \dfrac{\frac{1}{3}-1}{\frac{1}{3}+1} = \dfrac{-2/3}{4/3} = -\dfrac{1}{2}$.

$x=3$ $\quad f(4) = \dfrac{f(3)-1}{f(3)+1} = \dfrac{-\frac{1}{2}-1}{-\frac{1}{2}+1} = \dfrac{-3/2}{1/2} = -3$

$x=4$ $\quad f(5) = \dfrac{f(4)-1}{f(4)+1} = \dfrac{-3-1}{-3+1} = \dfrac{-4}{-2} = 2$

$f(1) = f(5) = f(9) = f(13) \ldots$

$f(2) = f(6) = f(10) \ldots$

$f(999) = f\left[\text{Rem}\left(\frac{999}{4}\right)\right]$

$= f(3) = -\frac{1}{2}$

If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, then function g such that $fog = h$ is -

(A) x + 1 (B) $x^2 + x - 1$ (C) $9x^2 + 9x + 11$ (D) none of these

$f\,[g(x)] = h(x)$

$g(x) = f^{-1}\,[h(x)]$

$f^{-1}(h(x)) = \dfrac{h(x)-5}{3}$

$= \dfrac{3x^2+3x+2-5}{}$

$g(x) = f^{-1}[h(x)]$

$y = f(x) = 3x+5$ swap x and y.

$x = 3y+5$

express y in terms of x.

$y = \frac{x-5}{3}$

$\downarrow$

$f^{-1}(x) = \frac{x-5}{3}$.

3.

$= \frac{3x^2+3x+2-5}{3}$

$= \frac{3x^2+3x-3}{3}$

$= x^2+x-1$

$$\lim_{x\to\infty}\left(\frac{3x^2+1}{3(x^2+x+1)}\right)^x =$$

(A) –1 (B) –2 (C) e^{-2} (D) e^{-1}

$y = \left(\frac{3x^2+1}{3x^2+3x+3}\right)^x$.

$\lim_{x\to\infty} y = ?$

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$

$\ln y = x \ln\left[\frac{3x^2+1}{3x^2+3x+3}\right]$

$\lim_{x\to a} \Rightarrow \frac{0}{0}$ form

or $\frac{\infty}{\infty}$ form.

$\Downarrow$

L'Hospital's Rule

$= \frac{f'(x)}{g'(x)}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left[\frac{3x^2+1}{3x^2+3x+3}\right]}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \left(\frac{3x^2+3x+3}{3x^2+1}\right) \frac{\frac{(3x^2+3x+3)6x-(3x^2+1)(6x+3)}{(3x^2+3x+3)^2}}{-\frac{1}{x^2}}$

$= \lim_{x\to\infty} - \frac{x^2\,\cancel{(3x^2+3x+3)}}{(3x^2+1)} \cdot \frac{\cancel{18x^3}+18x^2+18x-\cancel{18x^3}-9x^2-6x-3}{(3x^2+3x+3)^{\cancel{2}}}$

$= -\lim_{x\to\infty} \frac{1}{3+\frac{1}{x^2}} \cdot \frac{9x^2+12x-3}{3(x^2+x+1)} = -\lim_{x\to\infty} \frac{1}{3+\frac{1}{x^2}} \cdot \frac{3x^2+4x-1}{x^2+x+1}$

$= -\left(\lim_{x\to\infty} \frac{1}{3+\frac{1}{x^2}} \cdot \lim_{x\to\infty} \frac{3+4/x-1/x^2}{1+1/x+1}\right) = -\frac{1}{3}\times\frac{3}{1} = -1$

$\lim_{x\to\infty} \ln y = -1 \Rightarrow \ln\left(\lim_{x\to\infty} y\right) = -1 \Rightarrow \lim_{x\to\infty} y = e^{-1}$

Find all pairs (x, y) with x, y real, satisfying the equations:

$$\sin\left(\frac{x+y}{2}\right) = 0, \ |x| + |y| = 1.$$

$\sin\theta = 0 \Rightarrow \theta = n\pi$.

$(2n+1)\pi \ \sin x = 0$ $\quad$ $\sin x = 0 \ 2n\pi$

$\frac{x+y}{2} = n\pi$.

$\boxed{x+y = 2n\pi}$

$n=0 \Rightarrow x+y=0 \Rightarrow \boxed{x=-y}. \Rightarrow |x| = |y|$

$|x|+|y| = 1 \Rightarrow |x| = |y| = \frac{1}{2}. \Rightarrow$

$\boxed{x=\frac{1}{2},\ y=\frac{1}{2}}$ ✗

✓ $\boxed{x=-\frac{1}{2},\ y=\frac{1}{2}}$

✓ $\boxed{x=\frac{1}{2},\ y=-\frac{1}{2}}$

$\boxed{x=-\frac{1}{2},\ y=-\frac{1}{2}}$ ✗

$n \geq 1$. $\quad 1 = |x|+|y| \geq |x+y| = |2n\pi| = 2\pi|n| \geq 2\pi$.

$1 \geq 2\pi$. ✗

The complete solution of the equation $100x - 100[x] = 1$, where [.] = the greatest integer less than or equal to x, are -

gint.

(A) ✓ $x = n + \frac{1}{100}, n \in N$

(B) $x = n - \frac{1}{100}, n \in N$

(C) $x = n + \frac{1}{100}, n \in I$

(D) $n < x < n+1, n \in I$

$x = [x] + \{x\}$ where $\{x\} \Rightarrow$ fractional part of x.

$x - [x] = \{x\}$

$100(x - [x]) = 1$

$x - [x] = \frac{1}{100}$.

integer.

$x = [x] + \frac{1}{100} = n + \frac{1}{100}$.

$\lceil 1 \rceil + \lceil \sqrt{2} \rceil + \lceil \sqrt{3} \rceil + \ldots\ldots\ldots\ldots + \lceil \sqrt{10} \rceil =$

(A) 26 (B) 19 (C) 10 (D) none of these

$\sqrt{4} = 2.$ $\sqrt{9} = 3.$ $3 < \sqrt{10} < 4$

$[1] = 1$ $[\sqrt{2}] = 1$ $[\sqrt{3}] = 1$ $[\sqrt{4}] = 2.$ $[\sqrt{5}] = 2$

$[\sqrt{6}] = 2$ $[\sqrt{7}] = 2$ $[\sqrt{8}] = 2$ $[\sqrt{9}] = 3$ $[\sqrt{10}] = 3$