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(Random variables) and Probability Distributions.

Tossing two coins => Number of heads.

Ex: { HH → 2, HT → 1, TT → 0 }

real-numbers in a sample space is random variable.

① Random variable: is a real-valued function defined over a sample space.

S = { HH, HT, TT, ... } => { 2, 1, 0, ... } random-variable

S = { 1, 2, 3, 4, 5, 6 }   1+2=3, 2+3=5 ...

② Two types of Random variables:

a> **Discrete**: A rv which can assume finite or countable number of values.

b> **Continuous**: A rv which can take an uncountable infinite number of values.

③ **Probability Distribution**: A statement of all possible r.v or set of values of a r.v together with corresponding probabilities gives the probability distribution of a random variable.

Ex: Rolling of a dice.

X	P(X)

X	$P(x)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
	$\Sigma P(x) = 1$

$\Rightarrow$ Probability Distribution.

(4) Expectation of a random variable. $\Rightarrow$ $\boxed{E(X) = \sum_x x \cdot P(x)}$

(Mean or Average)

(5) Variance of a random variable $\Rightarrow$ $\boxed{V(X) = E\left[x - E(x)\right]^2}$

$$= E\left\{x^2 + E(x)^2 - 2xE(x)\right\}$$

$$= E(x^2) + 1\{E(x)\}^2 - 2E(x)E(x)$$

$$\boxed{V(x) = E(x^2) - \{E(x)\}^2}$$

Example1: Suppose a fair coin is tossed twice.
Let the number of heads obtained be (x).
Write down the probability distribution of X.
Calculate expectation $E(x)$ and variance $V(x)$ of X.

Solution: Probability Distribution of X (ie tossing a fair coin & obtaining head).

$S = \{HH, HT, TH, TT\}$

X	$P(x)$
0	1/4
	2/4 = 1/2

1/4

2/4

1/4 = 1/2

	1/4
0	2/4 = 1/2
1	1/4
2	
Total	$\sum P(x) = 1$

2/4 1/4
1/4 = 1/2

$$E(X) = \sum_x x \cdot P(x) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

$$\boxed{E(X) = 1}$$

$$V(X) = E(X^2) - \{E(X)\}^2$$

ie, $E(X^2) = \sum x^2 \cdot P(X=x) = 0^2 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4}$

$$E(X^2) = 0 + \frac{1}{2} + 1 = 3/2$$

$$\therefore V(X) = E(X^2) - \{E(X)\}^2$$

$$\boxed{V(X) = 3/2 - 1 = 1/2}$$

Probability Functions and Distribution Functions.

1. probability mass function (p.m.f)

For a discrete r.v X, there exists a function $f(x)$ such that

$$f(x) = P(X = x)$$

2. Probability Density function (p.d.f)

For a continuous r.v X, there exist a function such that for

$$\underline{f(x) = P(X=x)}$$

A function satisfying the following conditions follows p.m.f.

<a> $f(x) \geq 0$ for all values of x.

<b> $\sum_x f(x) = 1$

(*) $E(x) = \sum_x x \cdot f(x)$ and $V(x) = E(x^2) - \{E(x)\}^2$

such that for $a \leq b$

$$\boxed{\int_a^b f(x)\,dx = P(a \leq X \leq b)}$$

Conditions to satisfy p.d.f

(i) $f(x) \geq 0$ for all value of x

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow E(x) = \int x \cdot f(x)\,dx$ and so on.

Example 2: Can the function

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x=-1 \\ \frac{1}{4} & \text{for } x=0 \\ \frac{1}{2} & \text{for } x=1 \\ 0 & \text{elsewhere.} \end{cases}$$

be regarded as a probability function of some discrete r.v x? (i.e p.m.f) If yes, find mean and variance of x.

Solution: from the question, $f(x) \geq 0$ for all values of x,

(i) $f(-1) = \frac{1}{4} > 0$, $f(0) = \frac{1}{4} > 0$, $f(1) = \frac{1}{2} > 0$

(ii) $\sum_x f(x) = f(-1) + f(0) + f(1)$

$= \frac{1}{4} + \frac{1}{4} + \frac{1}{2}$

$= \frac{1}{2} + \frac{1}{2} = 1$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$\therefore \sum_x f(x) = 1$$

So, $f(x)$ can be regarded as a probability function with discrete r.v x which follows pmf.

Mean of r.v X, $E(X) = \sum_x x f(x)$

$$= -1 \times f(-1) + 0 \times f(0) + 1 \times f(1)$$

$$= -1 \times \frac{1}{4} + 0 \times \frac{1}{4} + 1 \times \frac{1}{2}$$

$$\boxed{E(X) = -1/4 + 1/2 = 1/4 \text{ (ans)}}$$

Variance of r.v X, $V(X) = E(x^2) - \{E(x)\}^2$

ie, $E(x^2) = (-1)^2 \times \frac{1}{4} + 0^2 \times \frac{1}{4} + 1^2 \times \frac{1}{2}$

$$\boxed{E(x^2) = \frac{1}{4} + \frac{1}{2} = \frac{1+2}{4} = \frac{3}{4}}$$

$$\therefore V(X) = E(x^2) - \{E(X)\}^2 = \frac{3}{4} - \left(\frac{1}{4}\right)^2 = \frac{3}{4} - \frac{1}{16}$$

$$\boxed{V(X) = \frac{11}{16}}$$

Example: The pmf $f(x)$ of a r.v X is zero, except at points $x = 0, 1, 2$

and $f(0) = c$

$f(1) = 2c - 3c^2$ at $c = 2 \Rightarrow \underline{f(1) = -8 < 0}$

$f(2) = 4c - 1$

$2 \times \frac{1}{3} - 3 \times \left(\frac{1}{3}\right)^2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

$\frac{2}{3} - \frac{3}{9}$

(i) Determine the value of c

(ii) find $E(X)$

$\frac{2}{3}$ • 9

(i) Determine the value of C

(ii) Find $P(X>0 / X<2)$

(iii) find $E(X)$ and $V(X)$.

Solution:

Since r.v X follows p.m.f

$\therefore \sum_x f(x) = 1$

or, $f(0)+f(1)+f(2) = 1$

or, $c + 2c - 3c^2 + 4c - 1 = 1$

or, $\boxed{3c^2 - 7c + 2 = 0}$

or, $3c^2 - c - 6c + 2 = 0$

or, $c(3c-1) - 2(3c-1) = 0$

or, $(c-2)(3c-1) = 0$

$\therefore \boxed{c = 2 \text{ or } 1/3}$

at $c=2$ $f(1) = -8 < 0$ $\therefore c=2$ is absurd

Hence value of $c = 1/3$ for which $f(x)>0$ for all x.

$\boxed{f(0) = 1/3,\ f(1) = 1/3 \text{ and } f(2) = 1/3}$

(ii) $P(\overbrace{X>0}^{A} / \overbrace{X<2}^{B}) = \dfrac{P\{X>0 \cap X<2\}}{P(X<2)}$

$= \dfrac{P(X=1)}{P(X=0)+P(X=1)}$

$= \dfrac{1/3}{1/3+1/3}$

$= \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{2}$

$= 1/2$ (ans).

two event A&B

$P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$X = 0, 1, 2$

(iii) $E(x) = \sum x f(x) = 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 2 \times \frac{1}{3} = \frac{3}{3} = 1$ (ans).

$V(x) = E(x^2) - \{E(x)\}^2 = \frac{5}{3} - 1^2 = \frac{5}{3} - 1 = 2/3$ (ans).

$E(x^2) = \sum x^2 f(x) = 0^2 \times \frac{1}{3} + 1^2 \times \frac{1}{3} + 2^2 \times \frac{1}{3} = \frac{5}{3}$