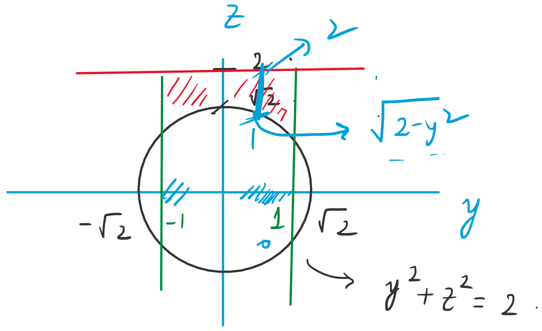
12 March 2024 21:09

Q. $\iiint_V dz\, dy\, dx$ where V is the region bounded by surfaces: $x=-1$, $x=1$, $y=-1$, $y=1$, $z=2$, $y^2+z^2=2$.



$$\int_{-1}^{1}\int_{-1}^{1}\int_{\sqrt{2-y^2}}^{2} dz\, dy\, dx$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[z\right]_{\sqrt{2-y^2}}^{2} dy\, dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} 2-\sqrt{2-y^2}\, dy\, dx$$

$$= \int_{-1}^{1} dx \left\{\int_{-1}^{1} 2-\sqrt{2-y^2}\, dy\right\}$$

$$= \int_{-1}^{1} dx \left\{\int_{-1}^{1} 2\,dy - \int_{-1}^{1}\sqrt{2-y^2}\, dy\right\}$$

use formula: $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

$$= \int_{-1}^{1} dx \underbrace{\left[2y-\left(\frac{y}{2}\sqrt{2-y^2}+\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)\right)\right]_{-1}^{1}}_{=\left(4-\frac{\pi}{2}\right)}$$

$$= \int_{-1}^{1}\left(4-\frac{\pi}{2}\right)dx = \left(4-\frac{\pi}{2}\right)\left[x\right]_{-1}^{1} = 2\left(4-\frac{\pi}{2}\right) = (8-\pi)$$

Q. $\iint_D \sqrt{x^2+y^2}\, dx\, dy$ where $D=\left\{(x,y)\in\mathbb{R}^2 \,/\, x\le x^2+y^2\le 2x\right\}$ is:

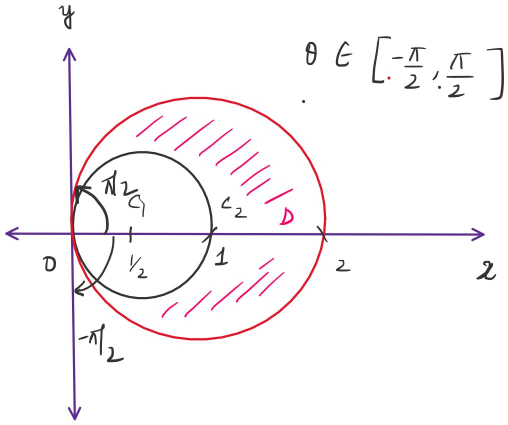
(a) $\frac{27}{8}$ (b) $\frac{28}{9}$ (c) $\frac{31}{4}$ (d) $\frac{36}{5}$

$D = \{(x, y) : x \le x^2 + y^2 \le 2x\}$ $\quad y^2 = x - x^2 \Rightarrow y = \sqrt{x - x^2}$

i.e $\quad x \le x^2 + y^2 \Rightarrow \boxed{x^2 + y^2 - x} \ge 0 \quad C\left(\frac{1}{2}, 0\right) \quad r = 1/2$ --- (i)

& $\quad x^2 + y^2 \le 2x \Rightarrow x^2 + y^2 - 2x \le 0 \quad C(1, 0) \quad r = 1$ --- (ii)

$$\iint_D \sqrt{x^2 + y^2}\, dy\, dx$$

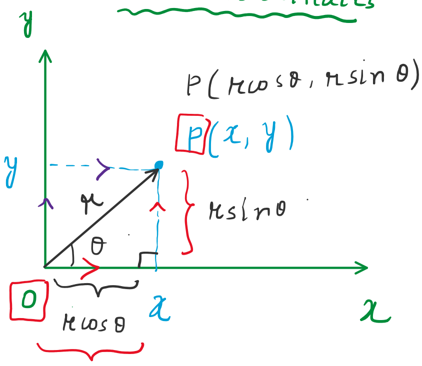


Transform into polar coordinate system: $(x, y) \to (r, \theta)$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$\iint dy\, dx = \iint dr\, d\theta.$$

Polar coordinates



Note: When we have a form like $x^2 + y^2$, then we'll use polar coordinates.

We have defined: $x = r\cos\theta$, $y = r\sin\theta$.

$$x^2 + y^2 = r^2$$

$$\frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

After defining r, θ, express x, y in terms of r, θ.

$$\cos\theta = \frac{B}{r} \Rightarrow B = r\cos\theta.$$

$$\sin\theta = \frac{P}{r} \Rightarrow P = r\sin\theta$$

$$(x, y) = (r\cos\theta, r\sin\theta)$$

$$x \le x^2 + y^2 \le 2x$$

$$r\cos\theta \le r^2 \le 2r\cos\theta.$$

$\boxed{\cos\theta \le r \le 2\cos\theta}$ $\Rightarrow$ range of 'r' dependent on θ.

$$\therefore I = \int_{-\pi/2}^{\pi/2} \int_{\cos\theta}^{2\cos\theta} f(r, \theta) |J|\, dr\, d\theta.$$

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$r \in [\cos\theta, 2\cos\theta]$

$-\pi/2 \quad \cos\theta$

$$I = \int_{-\pi/2}^{\pi/2} \int_{\cos\theta}^{2\cos\theta} r \cdot r \cdot dr\, d\theta .$$

$$= \int_{-\pi/2}^{\pi/2} \int_{\cos\theta}^{2\cos\theta} r^2\, dr\, d\theta .$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \left[\frac{r^3}{3}\right]_{\cos\theta}^{2\cos\theta}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{8\cos^3\theta - \cos^3\theta}{3}\, d\theta .$$

$$= \frac{1}{3} 7 \int_{-\pi/2}^{\pi/2} \cos^3\theta\, d\theta . \longrightarrow \text{Even fn}$$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta .$

$r \in [\cos\theta, 2\cos\theta]$

$f(x,y) = \sqrt{x^2+y^2}$

$f(r,\theta) = r .$

$(x,y) \to (r,\theta)$

$x = r\cos\theta ,$

$y = r\sin\theta$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r .$$

$\int f(x)\, dx$

$\int g(z)\, dz$

$z = \dots$

$dz = |J|\, dx .$

$$= \frac{2}{3} \cdot 7 \int_0^{\pi/2} \cos^3\theta\, d\theta .$$

$$= \frac{14}{3} \int_0^{\pi/2} \cos^2\theta \cdot \cos\theta\, d\theta .$$

$$= \frac{14}{3} \int_0^{\pi/2} (1-\sin^2\theta)\cos\theta\, d\theta .$$

$$= \frac{14}{3} \int_0^1 (1-z^2)\, dz$$

$\sin\theta = z .$

$\cos\theta\, d\theta = dz$

$\frac{d\theta}{dz} = \frac{1}{\cos\theta} .$

$$= \frac{14}{3}\left[z - \frac{z^3}{3}\right]_0^1$$

$$= \frac{14}{3}\left[1 - \frac{1}{3}\right] = \frac{14}{3} \cdot \frac{2}{3} = \frac{28}{9} .$$

Q. Let $I_1 = \iint_{D_1} dx\, dy \qquad D_1 = \{(x,y) : \sqrt{a} \le x \le 1,\ \sqrt{b} \le y \le 1\}$

$I_2 = \iint_{D_2} dx\, dy \qquad D_2 = \{(x,y) : ax^2 + by^2 \le 1\}$

Then: (a) $I_1 = I_2$ (b) $I_1 > I_2$ (c) $I_1 < I_2$ (d) None.