$$Var(X) = E(X^2) - [E(X)]^2$$
$$E(X^2) = Var(X) + [E(X)]^2$$

$$E(s^2) = E\left[\frac{1}{n}\sum x_i^2 - \bar{x}^2\right]$$

$$\Rightarrow E(s^2) = \frac{1}{n}\sum E(x_i^2) - E(\bar{x}^2) \quad \text{--- (1)}$$

$$\sigma^2 = E(x_i - \mu)^2 = E(x_i^2 - 2x_i\mu + \mu^2)$$

$$\Rightarrow \sigma^2 = E(x_i) - 2E(x_i)\mu + \mu^2$$

$$\Rightarrow \sigma^2 = E(x_i^2) - 2\mu^2 + \mu^2$$

$$\Rightarrow \sigma^2 = E(x_i^2) - \mu^2$$

$$\Rightarrow E(x_i^2) = \sigma^2 + \mu^2$$

$$v(\bar{x}) = \frac{\sigma^2}{n} = E(\bar{x}^2) - \{E(\bar{x})\}^2$$

$$\Rightarrow v(\bar{x}) = E(\bar{x}^2) - \mu^2 \quad (\because E(\bar{x}) = \mu)$$

$$\Rightarrow \frac{\sigma^2}{n} = E(\bar{x}^2) - \mu^2 \quad \rightarrow \text{Formula of variance.}$$

$$\Rightarrow \boxed{E(\bar{x}^2) = \frac{\sigma^2}{n} + \mu^2} \quad \text{--- (ii)}$$

(PTO)

$\sigma^2 = E(x_i - \mu$

$\Rightarrow \sigma^2 = E(x_i^2$

$\Rightarrow E(x_i^2) = \sigma$

$$Var(\bar{X}) = E(\bar{X}^2) - [E(\bar{X})]^2$$

$X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\begin{cases} E(\bar{X}) = \mu \\ Var(\bar{X}) = \frac{\sigma^2}{n} \end{cases}$$

$$E(\bar{X}^2) - [E(\bar{X})]^2 = \frac{\sigma^2}{n}$$

$$E(\bar{X}^2) - \mu^2 = \frac{\sigma^2}{n}$$

$$E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$$

Q. $X_1, X_2, \dots, X_n \overset{iid}{\sim} P(\lambda)$. Find the MLE of λ.

$X \sim P(\lambda)$.

pmf: $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, \dots$

$\therefore X_i \sim P(\lambda)\ \forall i$.

$$f(x_i) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}, \quad x_i = 0, 1, 2, \dots$$

Likelihood fn $L(\lambda) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

$$= \left(\frac{e^{-\lambda}\lambda^{x_1}}{x_1!}\right)\left(\frac{e^{-\lambda}\lambda^{x_2}}{x_2!}\right)\cdots\left(\frac{e^{-\lambda}\lambda^{x_n}}{x_n!}\right)$$

$$= \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod_{i=1}^{n}(x_i!)}$$

Log-likelihood fn: $\ell(\lambda) = -n\lambda + \sum x_i \ln\lambda - \ln\left(\prod_{i=1}^{n}(x_i)!\right)$

For MLE: $\dfrac{\partial \ell}{\partial \lambda} = 0 \Rightarrow -n + \dfrac{\sum x_i}{\lambda} = 0$.

$$\Rightarrow \frac{\sum x_i}{\lambda} = n \Rightarrow \hat{\lambda}_{MLE} = \frac{\sum x_i}{n} = \bar{x}$$

Q. $X_1, X_2, \dots, X_n \overset{iid}{\sim} Bin(m, p)$. Find the MLE of p.

$\therefore X_i \sim Bin(m,p) \ \forall i$

$$f(x_i) = {}^mC_{x_i}\, p^{x_i} (1-p)^{m-x_i}, \quad x_i = 0, 1, 2, \dots, m$$

$X \sim Bin(m, p)$

$f(x) = {}^mC_x\, p^x (1-p)^{m-x}, \ x = 0, 1, \dots$

$0 < p < 1$

Likelihood fn $L(p) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} {}^mC_{x_i}\, p^{x_i} (1-p)^{m-x_i}$

$$= \left\{ {}^mC_{x_1}\, p^{x_1} (1-p)^{m-x_1} \right\} \left\{ {}^mC_{x_2}\, p^{x_2} (1-p)^{m-x_2} \right\} \cdots \left\{ {}^mC_{x_n}\, p^{x_n} (1-p)^{m-x_n} \right\}$$

$$= \left(\prod_{i=1}^{n} {}^mC_{x_i} \right) p^{\sum x_i} (1-p)^{(m-x_1)+(m-x_2)+\dots+(m-x_n)}$$

$$= \left(\prod_{i=1}^{n} {}^mC_{x_i} \right) p^{\sum x_i} (1-p)^{mn - \sum x_i}$$

Log-likelihood fn: $\ell(p) = \ln\left(\prod_{i=1}^{n} {}^mC_{x_i} \right) + \sum x_i \ln p + (mn - \sum x_i) \ln(1-p)$

For MLE: $\dfrac{\partial \ell}{\partial p} = 0 \Rightarrow \dfrac{\sum x_i}{p} + \dfrac{(mn - \sum x_i)}{(1-p)}(-1) = 0$.

$$\Rightarrow \frac{\sum x_i}{p} = \frac{mn - \sum x_i}{(1-p)}$$

$$\Rightarrow \sum x_i - \cancel{p \sum x_i} = p \cdot mn - \cancel{p \sum x_i}$$

$$\Rightarrow \hat{p}_{MLE} = \frac{\sum x_i}{mn} = \frac{\bar{x}}{m}$$

Q. $\underbrace{X_1, X_2, \dots, X_n}_{\text{sample obs.}} \overset{iid}{\sim} \underbrace{N(\mu, \sigma^2)}_{\text{theoretical distribution.}}$. Find MoM Estimate of μ & σ^2.

$\rightarrow$ compute popln raw moments: $\mu_r' = E(X^r)$

[rth order popln raw moment] $= \int x^r f(x)\, dx$

$\downarrow$ compute sample raw moments: $m_r' = \frac{1}{n} \sum x_i^r$.

[rth order sample raw moment].

MoM: $\mu_r' = m_r'$, $r = 1, 2, \ldots$ [using these relations, find the estimator]

(1) For eg: for $X_1, X_2, \ldots, X_n \overset{iid}{\sim} P(\lambda)$

one-parameter, we'll use 1 moment eqn $\Rightarrow$ $m_1' = \mu_1'$

$$\Rightarrow \frac{1}{n}\sum x_i = E(X)$$

$$\Rightarrow \bar{x} = E(X) = \lambda.$$

$$\Rightarrow \hat{\lambda}_{MOM} = \bar{x}$$

Eg 2: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

2 parameters, we'll use 2 moment eqns.

$$m_1' = \mu_1'$$

$$\frac{1}{n}\sum x_i = E(X)$$

$$\bar{x} = \mu.$$

$$\therefore \boxed{\hat{\mu}_{MOM} = \bar{x}}$$

$$m_2' = \mu_2'$$

$$\frac{1}{n}\sum x_i^2 = E(X^2)$$ [By Formula].

$$\frac{1}{n}\sum x_i^2 = \underbrace{Var(X)}_{\sigma^2} + [\underbrace{E(X)}_{\mu}]^2$$

$$\frac{1}{n}\sum x_i^2 = \sigma^2 + \mu^2$$

$$\hat{\sigma}^2_{MOM} = \frac{1}{n}\sum x_i^2 - \hat{\mu}^2_{MOM}$$