17 February 2024 21:17

PRF (True Model) : $Y = X\beta + U$

SRF (Estimated Model) : $\hat{Y} = X\hat{\beta}$

set of NE [from OLS] is given by $e'X = \underset{\sim}{0}'$. $\Rightarrow$ Find $\hat{\beta}$

$e'X = \underset{\sim}{0}'$

Transpose : $X'e = \underset{\sim}{0}$ ----- (i)

Now $e = Y - \hat{Y} = Y - X\hat{\beta}$

$\therefore$ Replacing value of e in (i) : $X'(Y - X\hat{\beta}) = 0$.

$\Rightarrow X'Y - X'X\hat{\beta} = 0$.

$\Rightarrow (X'X)\hat{\beta} = X'Y$.

$\Rightarrow \boxed{\hat{\beta} = (X'X)^{-1}X'Y}$ $\Rightarrow$ OLS Estimates .

$(A \pm B)' = A' \pm B'$

$(AB)' = B'A'$

$(AB)^{-1} = B^{-1}A^{-1}$

$A^{-1} = \frac{1}{|A|} . Adj(A)$

$\rightarrow A(BC) = (AB)C$

But $AB \neq BA$.

$(A^{-1})' = (A')^{-1}$

Properties :

(i) $\hat{\beta} = (X'X)^{-1}X'Y$

$\hat{\beta} = (X'X)^{-1}X'(X\beta + U)$

$\hat{\beta} = (X'X)^{-1}(X'X)\beta + (X'X)^{-1}X'U$

$\hat{\beta} = \beta + (X'X)^{-1}X'U$. ---- [$\hat{\beta}$ is Linear in U] .

$\therefore E(\hat{\beta}) = \beta + (X'X)^{-1}X' \underbrace{E(U)}_{0}$. [$\because X$ is non-stochastic]

$E(\hat{\beta}) = \beta$ $\Rightarrow \hat{\beta}$ is an unbiased estimator of β .

(ii) $Var(\hat{\beta}) = E\{ [\hat{\beta} - E(\hat{\beta})][\hat{\beta} - E(\hat{\beta})]' \}$

$= E\{ [\hat{\beta} - \beta][\hat{\beta} - \beta]' \}$

$\{(X'X)^{-1}\}'$

$\{\boxed{(X'X)'}\}^{-1}$

$(X'X)^{-1}$

$$= E\left\{\left[(X'X)^{-1}X'U\right]\left[(X'X)^{-1}X'U\right]'\right\} \qquad (\hat{\beta}-\beta) = (X'X)^{-1}X'U$$

$$= E\left\{(X'X)^{-1}X'(UU')X(X'X)^{-1}\right\}$$

$$= (X'X)^{-1}X'E(UU')X(X'X)^{-1}$$

$$= (X'X)^{-1}X'(\sigma^2 I)X(X'X)^{-1}$$

$$= \sigma^2 (X'X)^{-1}(X'X)(X'X)^{-1}$$

$$\boxed{Var(\hat{\beta}) = \sigma^2 (X'X)^{-1}}$$

Note: As σ^2 is unknown popln parameter, for reporting purposes we need to replace σ^2 by its estimate $\hat{\sigma}^2$.

Result: $E(\sum e_i^2) = (n-k)\sigma^2$

$$\Rightarrow E\left(\underbrace{\frac{\sum e_i^2}{n-k}}_{\hat{\sigma}^2}\right) = \sigma^2 \Rightarrow E(\hat{\sigma}^2) = \sigma^2 \text{ where } \hat{\sigma}^2 = \frac{\sum e_i^2}{(n-k)}$$

$$\therefore \widehat{Var}(\hat{\beta}) = \hat{\sigma}^2 (X'X)^{-1} \qquad [\hat{\sigma}^2 \text{ is defined}]$$

$$s.e(\hat{\beta}) = \sqrt{\widehat{Var}(\hat{\beta})} = \sqrt{\hat{\sigma}^2 (X'X)^{-1}}$$