Topic: Maxima and Minima: case of several variables. [Unconstrained Optimisation].

Questions:

1. Find the stationary values and test whether they are maximum or minimum for ($z = 3x^2 + 6xy + 7y^2$)

2. Find the stationary values and examine their nature for $z = 5x^2 + 3y^2 - 15xy$.

3. Show that the only stationary point the function $z = 4x^2 - 2y^2 + 7xy$ has is a saddle point.

4. The revenue function of a firm is given by
$$R = 50x_1 + 500x_2 - x_1^2 - x_2^2 - x_1x_2$$
where x_1 and x_2 are product of two different cost functions, $C_1 = 3x_1^2 + 33$
$C_2 = 4x_2^2 + 44$
Find maximum profit firm can make.
max π = ? → x_1 and x_2

5. Let cost fn be $C = 10 + 15x$ of a firm that faces demand curves $P_1 = 55 - 2x_1$
$P_2 = 25 - 5x_2$
in two markets. Find x_1 and x_2 that max profit. Also find P_1 and P_2.

single variable fn → $y = f(x)$
one indp variable.

... variable fn ... $f(x_1, x_2)$

several variable fn (more than one indp variable) $\Rightarrow$ $\boxed{y = f(x_1, x_2)}$

	Maximisation	Minimisation.
F.O.C (Necessary)	$\frac{\partial y}{\partial x_1}$ or $f_1 = 0$ $\frac{\partial y}{\partial x_2}$ or $f_2 = 0$	$f_1 = 0$ $f_2 = 0$
S.O.C (Sufficiency)	$\frac{\partial^2 y}{\partial x_1^2}$ or $f_{11} < 0$ $\frac{\partial^2 y}{\partial x_2^2}$ or $f_{22} < 0$ $f_{11} f_{22} > (f_{12})^2$	$f_{11} > 0$ $f_{22} > 0$ $f_{11} f_{22} > (f_{12})^2$

special condition:

a) a saddle point if $f_{11} f_{22} < (f_{12})^2$

and f_{11} and f_{22} have opp signs (or different signs)

[if $f_{11} > 0$ then $f_{22} < 0$ | or $f_{11} < 0$ then $f_{22} > 0$]

b) a point of inflexion if $f_{11} f_{22} < (f_{12})^2$

and f_{11} and f_{22} have same sign.

sign.

SOLUTIONS :

1) We are given, $Z = 3x^2 + 6xy + 7y^2$

F.O.Cs $f_x = \frac{\partial z}{\partial x} = 6x + 6y = 0$ — ① $f_y = \frac{\partial z}{\partial y} = 6x + 14y = 0$ — ②

Solving ① and ②,

$$\begin{array}{r} \cancel{6x} + 6y = 0 \\ - \cancel{6x} + 14y = 0 \\ \hline -8y = 0 \end{array}$$

$\Rightarrow \boxed{y = 0}$

$\therefore \boxed{x = 0}$

To check S.O.C, we have to find,

$f_{xx} = \frac{\partial^2 z}{\partial x^2} = 6 > 0$ $f_{yy} = \frac{\partial^2 z}{\partial y^2} = 14 > 0$ $f_{xy} = 6$

$f_{xy}^2 = 36$

Now. $f_{xx} \cdot f_{yy} = 6 \times 14 = 84 > 36 (= f_{xy}^2)$

$\therefore$ at $x = 0, y = 0 \Rightarrow z$ is minimised.

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2) $Z = 5x^2 + 3y^2 - 15xy$

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$$Z = 5x^2 + 3y^2 - 15xy$$

F.O.C

$f_x = \frac{\partial z}{\partial x} = 0$

$\Rightarrow 10x - 15y = 0 \quad \text{— (1)} \times 2.$

$f_y = \frac{\partial z}{\partial y} = 0$

$6y - 15x = 0 \quad \text{— (2)} \times 5$

Solving (1) and (2)

$$\begin{array}{l} 20x - \cancel{30y} = 0 \\ -75x + \cancel{30y} = 0 \\ \hline -55x = 0 \end{array}$$

$\boxed{x = 0}$ and $\boxed{y = 0}$

Stationary pts are $(0, 0)$.

To check the nature we have to calculate,

$f_{xx} = \partial^2 z / \partial x^2 = 10 > 0$

$f_{yy} = \partial^2 z / \partial y^2 = 6 > 0$

$f_{xy} = -15$

$f_{xy}^2 = 225$

$f_{xx} \cdot f_{yy} = 60$

$\therefore f_{xx} \cdot f_{yy} < f_{xy}^2 \quad [\because 60 < 225]$

Since f_{xx} and f_{yy} have same sign and $f_{xx} f_{yy} < f_{xy}^2$

... the function is that at pt $(0,0)$

Since f_{xx} and f_{yy} have same sign ...

$\therefore$ The nature of the function is that at pt $(0,0)$ the fn will have inflexion point. (Ans)

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(3) $z = 4x^2 - 2y^2 + 7xy$

F.O.C

$f_x = \frac{\partial z}{\partial x} = 0$

$\Rightarrow 8x + 7y = 0$ — (1)

$f_y = \frac{\partial z}{\partial y} = 0$

$\Rightarrow -4y + 7x = 0$ — (2)

solving (1) and (2), we get $(x = 0$ and $y = 0)$

S.O.C

$f_{xx} = \frac{\partial^2 z}{\partial x^2} = 8 > 0$

$f_{yy} = \frac{\partial^2 z}{\partial y^2} = -4 < 0$

$f_{xy} = 7$

$f_{xy}^2 = 49$

$f_{xx} \cdot f_{yy} = -32$

Since $-32 < 49$

$\therefore f_{xx} \cdot f_{yy} < f_{xy}^2$

Since f_{xx} and f_{yy} have opposite signs and $f_{xx} f_{yy} < f_{xy}^2$,

$\therefore$ fn z has a saddle point.

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(4) Given: $R = 50x_1 + 500x_2 - x_1^2 - x_2^2 - x_1x_2$

$C_1 = 3x_1^2 + 33 \qquad C_2 = 4x_2^2 + 44.$

$TC = C_1 + C_2$

Profit function, $\pi = TR - TC$

$= (50x_1 + 500x_2 - x_1^2 - x_2^2 - x_1x_2) - (3x_1^2 + 33 + 4x_2^2 + 44)$

$$\boxed{\pi = -4x_1^2 - 5x_2^2 + 50x_1 + 500x_2 - x_1x_2 - 77} \quad \text{— (1)}$$

For profit maximisation,

FOC $\Rightarrow \dfrac{\partial\pi}{\partial x_1} = \pi_1 = 0 \Rightarrow -8x_1 + 50 - x_2 = 0$

$\Rightarrow 8x_1 + x_2 = 50$ — (2) $\times 10$

$\partial\pi/\partial x_2 = \pi_2 = 0 \Rightarrow -10x_2 + 500 - x_1 = 0$

$\Rightarrow x_1 + 10x_2 = 500$ — (3)

Solving (2) and (3):

$$\begin{array}{rcl} 80x_1 + \cancel{10x_2} &=& 500 \\ -\,x_1 + \cancel{10x_2} &=& 500 \\ \hline 79x_1 &=& 0 \end{array}$$

$$\boxed{x_1 = 0}$$

from (3) $\Rightarrow \boxed{x_2 = \dfrac{500}{10} = 50}$

S.O.C — | $\pi_{..} = -1$

S.O.C

$\pi_{11} = -8 < 0$

$\pi_{22} = -10 < 0$

$\pi_{11}\pi_{22} = 80$

$\pi_{12} = -1$

$\pi_{12}^2 = 1$

$\because 80 > 1$

$\therefore \pi_{11}\pi_{22} > \pi_{12}^2$

$\therefore$ at $x_1 = 0$ and $x_2 = 50$ π is maximised.

$\therefore \max\pi = \cancel{50x_1} + 500x_2 - \cancel{x_1^2} - x_2^2 - \cancel{x_1x_2} - \cancel{3x_1^2} - 4x_2^2 - 77$

$= 12423$ (ans)

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Total cost of $x \to x_1, x_2$

(5) $C = 10 + 15x$

$x = x_1 + x_2$

$P_1 = 55 - 2x_1$

$TR_1 = P_1 \cdot x_1 = 55x_1 - 2x_1^2$

$P_2 = 25 - 5x_2$

$TR_2 = P_2 \cdot x_2 = 25x_2 - 5x_2^2$

$TR = TR_1 + TR_2$

$\pi = TR - TC$

$= (55x_1 - 2x_1^2 + 25x_2 - 5x_2^2) - (10 + 15x_1 + 15x_2)$

$+ 25x_2 - 5x_2$

$$\pi = -2x_1^2 - 5x_2^2 + 40x_1 + 10x_2 - 10 \longrightarrow (1)$$

To maximise π $\Rightarrow$ f.o.c requires,

$\pi_1 = 0 \Rightarrow -4x_1 + 40 = 0$

$\Rightarrow \boxed{x_1 = 10}$

$\pi_2 = 0 \Rightarrow -10x_2 + 10 = 0$

$\boxed{x_2 = 1}$

S.O.C :

$\pi_{11} = -4 < 0$

$\pi_{22} = -10 < 0$

$\pi_{12} = 0$

$\pi_{11} \cdot \pi_{22} = 40$

$\therefore \pi_{11}\pi_{22} > \pi_{12}^2$

$\therefore$ At $x_1 = 10$ and $x_2 = 1 \Rightarrow \pi$ is maximised.

$\therefore P_1 = 55 - 2x_1 = 55 - 2\times10$

$= 55 - 20$

$\boxed{P_1 = 35}$

$P_2 = 25 - 5x_2$

$= 25 - 5$

$\boxed{P_2 = 20}$

(ans).