Quadratic Inequalities

Illustration 6.38 If α and β are the roots of the equation $(a-2)x^2-(5-a)x-5=0$. Find a if $|\alpha-\beta|=2\sqrt{6}$.

$$(\alpha-\beta)^2 = 24 = (\alpha+\beta)^2 - 4\alpha\beta$$
$$= \left(\frac{5-a}{a-2}\right)^2 + \frac{20}{a-2}$$

$a = 3$ or $37/23$

If α and β are roots of the equation $ax^2+bx+c=0$, then the equation whose roots are

1. $-\alpha, -\beta$ is $ax^2-bx+c=0$ (Replace x by $-x$)
2. $1/\alpha, 1/\beta$ is $cx^2+bx+a=0$ (Replace x by $1/x$)
3. $\alpha^n, \beta^n; n \in N$ is $a(x^{1/n})^2+b(x^{1/n})+c=0$ (Replace x by $x^{1/n}$)
4. $k\alpha, k\beta$ is $ax^2+kbx+k^2c=0$ (Replace x by x/k)
5. $k+\alpha, k+\beta$ is $a(x-k)^2+b(x-k)+c=0$ [Replace x by $(x-k)$]
6. $\frac{\alpha}{k}, \frac{\beta}{k}$ is $k^2ax^2+kbx+c=0$ (Replace x by kx)
7. $\alpha^{1/n}, \beta^{1/n}; n \in N$ is $a(x^n)^2+b(x^n)+c=0$ (Replace x by x^n).

Interval satisfied by $\frac{x^2-8x+12}{x-6} > 0$

$$\frac{(x-6)(x-2)}{(x-6)} > 0$$

$x \in (2,6) \cup (6,\infty)$

$\in (2,\infty) - \{6\}$

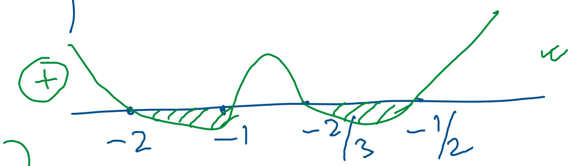
$$\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$$

$\Rightarrow \frac{2x}{(2x+1)(x+2)} - \frac{1}{x+1} > 0$

$\Rightarrow \frac{2x(x+1) - 2x^2-5x-2}{(x+1)(x+2)(2x+1)} > 0$

$\Rightarrow \frac{-3x-2}{(x+1)(x+2)(2x+1)} > 0$

$\Rightarrow \frac{x+2/3}{(x+1)(x+2)(2x+1)} < 0$



$x = -1, -2, -1/2, -2/3$

Soln $x \in (-2,-1) \cup (-2/3, -1/2)$

$x^2-3x+2 > 0$ & $x^2-3x-4 \le 0$

$(x-1)(x-2) > 0$

$x = 1, 2$

$x \in (-\infty, 1) \cup (2, \infty)$ ← Intersection ⇒ $x \in [-1, 4]$

$(x+1)(x-4) \le 0$

$x = -1, 4$

HW (1) $\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3$ (2) $x^2 - |x+2| + x > 0$

Binomial Theorem

$(n-1)$ terms

$x^n + \dots + a^n$

1. There are $(n+1)$ terms in the expansion of $(x+a)^n$.
2. Sum of powers of x and a in each term in the expansion of $(x+a)^n$ is constant and is equal to n.
3. The p^{th} term from the end $=(n-p+2)^{th}$ term from the beginning.
4. Coefficient of x^{n-r} in the expansion of $(x+a)^n$ is ${}^nC_r \cancel{x^{n-r}a^r}$.
5. ${}^nC_x = {}^nC_y \Rightarrow x=y$ or $x+y=n$.
6. In the expansion of $(x+a)^n$ and $(x-a)^n$, x^r occurs in $(r+1)^{th}$ term.
7. If n is odd, then $(x+a)^n+(x-a)^n$ and $(x+a)^n-(x-a)^n$, both have the same number of terms, that is, $\left(\frac{n+1}{2}\right)$.
8. If n is even, then $(x+a)^n+(x-a)^n$ has $\left(\frac{n}{2}+1\right)$ terms and $(x+a)^n-(x-a)^n$ has $\frac{n}{2}$ terms.
9. The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is ${}^nC_r$.
10. The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

$(1+x)^n = 1 + C_1 + C_2x^2 + C_3x^3$

If coefficients of 2^{nd}, 3^{rd} & 4^{th} term of $(1+x)^n$ are in AP, find n

${}^nC_1, {}^nC_2, {}^nC_3$ $\qquad 2\,{}^nC_2 = {}^nC_1 + {}^nC_3$

$$\Rightarrow 2\frac{n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$$

$$\Rightarrow n^2-9n+14=0 \Rightarrow (n-2)(n-7)=0 \Rightarrow n=2 \text{ or } 7$$

Find the last 2 & 3 digits of 17^{2024}

$$(17)^{2024} = (289)^{1012} = (290-1)^{1012} = (1-290)^{1012}$$

$$(1-290)^{1012} = 1 - {}^{1012}C_1 \cdot 290 + {}^{1012}C_2(290)^2 - $$

$$= 1 - 1012\times 290 + \frac{1012\times 1011}{2}\times(290)^2$$

HW Find 'x' independent term : $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$

$$T_{r+1} = {}^{10}C_r\left(\sqrt{\frac{x}{3}}\right)^{10-r}\left(\frac{3}{2x^2}\right)^r = {}^{10}C_r\left(\frac{1}{\sqrt3}\right)^{10-r} x^{\frac{10-r}{2}-2r}\left(\frac{3}{2}\right)^r$$

$$\frac{10-r}{2} = 2r \Rightarrow 10 = 5r \Rightarrow r=2$$

$$T_{2+1} = {}^{10}C_2\left(\frac{1}{\sqrt3}\right)^8\left(\frac{3}{2}\right)^2 = \frac{5}{4}$$

Binomial Theorem

$$\left(3x - \frac{x^3}{9}\right)^9 \qquad T_{4+1} = {}^9C_4 (3x)^5 \left(-\frac{x^3}{9}\right)^4 = \frac{189}{8} x^{17}$$

$$T_{5+1} = {}^9C_5 (3x)^4 \left(-\frac{x^3}{9}\right)^5 = -\frac{21}{16} x^{19}$$

$$\left(x + \frac{1}{x}\right)^{10} \rightarrow T_6 \rightarrow {}^{10}C_5 \qquad (1-2x+x^2)^n$$

$$(1-x)^{2n} \rightarrow {}^{2n}C_n (-x)^n$$

Numerically greatest term : $\dfrac{T_{r+1}}{T_r} = \dfrac{{}^nC_r x^r}{{}^nC_{r-1} x^{r-1}} = \dfrac{n-r+1}{r} \cdot x$

$$\frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{n-r+1}{r}|x| \geq 1 \Rightarrow r \leq \frac{(n+1)|x|}{(1+|x|)}$$

(1) $m = \dfrac{(n+1)|x|}{1+|x|}$

(A) $m \in \mathbb{Z}$, $T_m, T_{m+1} \rightarrow$ equal & greatest

(B) If not, $T_{[m]+1} \rightarrow$ greatest

$(3+2x)^{50}$, with $x = 1/5$

$$3^{50}\left(1 + \frac{2}{3}x\right)^{50} \qquad m = \frac{2x/3 \cdot 51}{1 + 2x/3} = \frac{34x \cdot 3}{3+2x} = 6$$

T_6 & T_7 are equal & greatest terms

$(4+3x)^7$, with $x = 2/3$

$$\frac{T_{r+1}}{T_r} = \frac{{}^7C_r \, 4^{7-r} \cdot 3x^r}{{}^7C_{r-1} \, 4^{8-r} \quad 3x^{r-1}} = \frac{8-r}{r} \cdot \frac{3x}{4} = \frac{8-r}{2r}$$

$\dfrac{T_{r+1}}{T_r} \geq 1$ if $8 - r \geq 2r$ $\therefore r \leq 8/3$

$T_1 < T_2 < T_3 > T_4 > T_5 >$

$$T_3 = {}^7C_2 \, 4^5 \cdot (3x)^2 = 21 \times 4^6$$
$$= 86016$$

1. ${}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$
2. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
3. $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$
4. $\dfrac{{}^nC_r}{r+1} = \dfrac{{}^{n+1}C_{r+1}}{n+1}$

Sum of binomial coefficients

$$(1+x)^n = 1 + C_1 x + C_2 x^2 + \cdots$$

Put $x = 1$, $2^n = C_0 + C_1 + C_2 + \cdots$

Put $x = -1$, $0 = C_0 - C_1 + C_2 - C_3 + \cdots$

$\Rightarrow C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 \cdots$

Sequences and Series

Find ${}^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$ Relate to $(1+x)^8$, ans 2^{8-1}

(*) Evaluate $C_0 - C_2 + C_4 - C_6 + \cdots$ (HW)

(*) **Illustration 8.40** Show that

$$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \cdots + C_{n-r}C_n = \frac{(2n)!}{(n-r)!(n+r)!}$$

Illustration 8.37 Show that

1. $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$
2. $3 \cdot C_0 + 3^2 \cdot \frac{C_1}{2} + 3^3 \cdot \frac{C_2}{3} + \cdots + 3^{n+1}\frac{C_n}{n+1} = \frac{4^{(n+1)}-1}{n+1}$
3. $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + \cdots = \frac{1}{(n+1)(n+2)}$

$\int_0^1 (1+x)^n$ $\int_0^3 (1+x)^n$ $\int_0^1 x(1-x)^n$

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r + \cdots$
2. $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}(-x)^r + \cdots$
3. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots + \frac{n(n+1)\cdots(n+r-1)}{r!}x^r + \cdots$
4. $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots + \frac{n(n+1)\cdots(n+r-1)}{r!}(-x)^r + \cdots$

If $f(n) = \frac{n}{(2n+1)^2}$, write the first 3 terms in ordered format

$f(1) = \frac{1}{9}$, $f(2) = \frac{2}{25}$

$f(3) = \frac{3}{49}$

pairs $= \{(1, 1/9), (2, 2/25), (3, 3/49)\}$

$\hookrightarrow (x, f(x))$

<u>Common diff</u>

$T_{n+1} - T_n = T_n - T_{n-1} = d$

If a, b, c are x, y, z^{th} terms of an AP, show

(1) $a(y-z) + b(z-x) + c(x-y) = 0$

$T_x = A + (x-1)D$, T_y, T_z — (1, 2, 3)

Subtract pairwise: $(a-b) = (x-y)D \Rightarrow (x-y) = \frac{a-b}{D}$

$y - z = \frac{b-c}{D}$, $z - x = \frac{c-a}{D}$ Put in LHS

(2) $x(b-c) + y(c-a) + z(a-b) = 0$

Use same transformation