Random Variables (1)

For doubts please contact : 7890368395 / 8017896203

25 May 2023 18:48

Variable: Height of an individual in a classroom 10 students.

fixed

Random Variable:
- Probability is associated with it.
- the set of all possible outcomes are known.
- it has a distribution.

X : No. of heads obtained in 2 tosses of an unbiased coin.

E : Tossing 2 coins. $\longrightarrow$ Experiment

$$\Omega = \underbrace{\{HH, HT, TH, TT\}}$$

$\underbrace{\text{No. of heads}}_{X}$
- 0 when $\{TT\}$ occurs.
- 1 when $\{HT, TH\}$ occurs.
- 2 when $\{HH\}$ occurs.

$$X = \begin{cases} 0 & \text{w.p } 1/4 \\ 1 & \text{w.p } \frac{1}{4}+\frac{1}{4} \\ 2 & \text{w.p } \frac{1}{4} \end{cases}$$

Total probability $= \frac{1}{4}+\left(\frac{1}{4}+\frac{1}{4}\right)+\frac{1}{4}$

$= 1$

Classification
- Discrete RV (Distribution: pmf)
- Continuous RV (Distribution: pdf)

Classification ← Continuous RV (Distribution: pdf)

Properties of Probability Distribution:

For any discrete r.v X,

$f(x) = P(X = x)$ = probability mass function of X

$\underset{RV}{X}$

i) $f(x) \geq 0 \quad \forall x$.

ii) $\sum_x f(x) = 1$.

eg: 1) $X \sim Bin(n, p)$,

$$f(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & , x = 0, 1, \ldots, n. \\ 0 & , \text{o.w} \end{cases}$$

where, $0 < p < 1$ and $p + q = 1$.

Check whether $f(x)$ is really a pmf or not.

Ans: Clearly $f(x) \geq 0 \quad \forall x = 0, 1, \ldots, n$.

$$\sum_x f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

$$= \binom{n}{0} p^0 q^{n-0} + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2}$$

$$= \binom{n}{0} p^0 q^{n-0} + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots + \binom{n}{n} p^n q^{n-n}$$

$$= q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots + p^n$$

$$= (q+p)^n = (1)^n = 1$$

2) Three balls are drawn at random from a bag containing 5 white and 3 black balls. If X denote the no. of white balls drawn,

1 (Write down the probability distribution of X.) 2 (Verify whether it is really a pmf or not). (Also obtain $E(x)$ and $V(x)$) 3.

5 W
3 B

$\xrightarrow{\text{at random}}$ 3 balls

Bag = 8 balls

5 + 3

X : The no. of white balls drawn.

Out of 8 balls 3 balls are randomly selected

$$X = 0, 1, 2, 3$$

$$P(X=0) = \frac{\binom{3}{3}}{\binom{8}{3}} = \frac{1}{56}.$$

None of the

$\binom{}{3}$

None of the white ball is chosen

$$P(X=1) = \frac{\binom{5}{1}\times\binom{3}{2}}{\binom{8}{3}} = \frac{15}{56}$$

One white ball is chosen

$$P(X=2) = \frac{\binom{5}{2}\times\binom{3}{1}}{\binom{8}{3}} = \frac{30}{56}$$

Two white balls are chosen

$$P(X=3) = \frac{\binom{5}{3}}{\binom{8}{3}} = \frac{10}{56}$$

All the balls are white.

$\therefore$, The probability distribution of X is given by

$$X = \begin{cases} 0 & \text{w.p. } 1/56 \\ 1 & \text{w.p. } 15/56 \\ 2 & \text{w.p. } 30/56. \\ 3 & \text{w.p. } 10/56 \end{cases}$$

$$f(0) = P(X=0) = \frac{1}{56} > 0$$

$$f(1) = P(X=1) = 15/56 > 0$$

$$f(1) = P(X=1) = 15/56 > 0$$

$$f(2) = P(X=2) = 30/56 > 0$$

$$f(3) = P(X=3) = 10/56 > 0$$

$$\sum_x f(x) = \sum_{x=0}^{3} = f(0) + f(1) + f(2) + f(3)$$

$$= \frac{1}{56} + \frac{15}{56} + \frac{30}{56} + \frac{10}{56}$$

$$= 1$$

Yes, it is a pmf.

$$E(X) = \sum_x x\,f(x) = \sum_{x=0}^{3} x\,f(x)$$

$$= 0.f(0) + 1.f(1) + 2.f(2) + 3.f(3)$$

$$= 0 + \frac{15}{56} + 2 \times \frac{30}{56} + 3 \times \frac{10}{56}$$

$$= \frac{15+60+30}{56} = 15/8 = 1\frac{7}{8}.$$

$$E(X^2) = \sum_{x=0}^{3} x^2 f(x) = 0^2 \times f(0) + 1^2 \times f(1) + 2^2 \times f(2) + 3^2 \times f(3)$$

$$= 0 + 1 \times \frac{15}{56} + 4 \times \frac{30}{56} + 9 \times \frac{10}{56}$$

$$\vdots$$

$$= \frac{225}{56}$$

$$= \frac{225}{56}$$

$$V(X) = E(X^2) - E^2(X) = \frac{225}{448}.$$

For any continuous r.v X, the pdf is given by, $f(x)$, which is called probability density function.

<u>Properties:</u> 1) $f(x) \geq 0 \ \forall x$.

2) $\int_x f(x)\,dx = 1$.

<u>eg ①</u> $X \sim Exp(mean = \frac{1}{\lambda})$.

Check: $f(x) = \lambda e^{-\lambda x}, x > 0$ is a pdf or not.

<u>Ans:</u> Clearly, $f(x) \geq 0 \ \forall x > 0$

$$\int_x f(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}\,dx$$

$\lambda x = p$

$\Rightarrow \lambda dx = dp$

x	0	∞
p	0	∞

$$= \cancel{\lambda}\int_0^\infty e^{-p}\,\frac{dp}{\cancel{\lambda}}$$

$$= \int_0^\infty e^{-p}\,dp$$

$$-\int_0 e^{-p}\, dp$$

$$= \left[-e^{-p}\right]_0^{\infty} = -(0-1) = 1$$

2) Consider, $f(x) = \begin{cases} Kx, & 0 < x \leq 1 \\ 4-2x, & 1 < x \leq 2 \\ 0 & , \text{o.w.} \end{cases}$

a) Find K

b) Find $E(x)$ and $V(x)$.

Ans: $\int_x f(x)\, dx = 1$

$$\Rightarrow 1 = \int_0^1 2x\, dx + \int_1^2 (4-2x)\, dx$$

$$\Rightarrow 1 = K\left[\frac{x^2}{2}\right]_0^1 + 4\left[x\right]_1^2 - 2\left[\frac{x^2}{2}\right]_1^2$$

$$\Rightarrow 1 = (1-0)K + 4(2-1) - (4-1)$$

$$\Rightarrow 1 = K + 4 - 3$$

$$\Rightarrow K + 1 = 1$$

$\Rightarrow \quad K = 0$

Now, $f(x) = \begin{cases} 0 \cdot x, & 0 < x < 1 \\ 4 - 2x, & 1 < x < 2 \\ 0, & \text{o.w} \end{cases}$

$= \begin{cases} 4 - 2x, & 1 < x < 2 \\ 0, & \text{o.w} \end{cases}$

Now, $\int_x f(x)\,dx = \int_1^2 (4-2x)\,dx = 4[x]_1^2 - 2\left[\frac{x^2}{2}\right]_1^2$

$= 4(2-1) - (4-1)$

$= 4 - 3 = 1$

Hence, for $k = 0$, it is a pdf.

$E(X) = \int_1^2 x f(x)\,dx = \int_1^2 x(4-2x)\,dx.$

$E(X^2) = \int_1^2 x^2 f(x)\,dx = \int_1^2 x^2(4-2x)\,dx$

$\therefore, \quad V(X) = E(X^2) - E^2(X).$