Utility Maximization

Given a utility fn: $u = u(x_1, x_2)$.

Have a budget constraint: $M = P_1 x_1 + P_2 x_2$

} Parameters: M, P_1, P_2.
Variables: x_1, x_2.

Objective of the consumer: Maximize utility subject to Budget constraint.

$$\max_{\{x_1, x_2\}} u = u(x_1, x_2) \quad \text{subject to} \quad M = P_1 x_1 + P_2 x_2 .$$

[Assume $u(\cdot)$ is differentiable].

Define the Lagrangian as:

$$\mathcal{L} = u(x_1, x_2) + \lambda [M - P_1 x_1 - P_2 x_2] \; ; \; \lambda = \text{Lagrange multiplier}$$

Graphically,
At optimal;

$$\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \Rightarrow \frac{\partial u}{\partial x_1} + \lambda(-P_1) = 0 \Rightarrow MU_1 = \lambda P_1 .$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \Rightarrow \frac{\partial u}{\partial x_2} + \lambda(-P_2) = 0 \Rightarrow MU_2 = \lambda P_2 .$$

$$\Rightarrow \frac{MU_1}{MU_2} = \frac{P_1}{P_2} \quad ✓$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow M - P_1 x_1 - P_2 x_2 = 0 \Rightarrow M = P_1 x_1 + P_2 x_2 .$$

We obtained the Marshallian demand curves:

$$x_1^* = x_1^*(P_1, P_2, M) \qquad x_2^* = x_2^*(P_1, P_2, M)$$

↳ Given P_1, P_2, M, what is the utility maximizing consumption level of Good 1.

Now: $u^* = u(x_1^*, x_2^*) \Rightarrow$ Maximized value of utility.

$$u^* = u(x_1^*(\underbrace{P_1, P_2, M}), x_2^*(\underbrace{P_1, P_2, M}))$$

$$= u^*(P_1, P_2, M) \Rightarrow$$ Maximized value of utility given P_1, P_2, M.

$$= V(P_1, P_2, M)$$

↳ Indirect

$= V(P_1, P_2, M)$ ...

↳ Indirect utility Function

Given: $u = u(x_1, x_2) \longrightarrow V = V(P_1, P_2, M)$.

Q. $u(x_1, x_2) = x_1^a x_2^b$. Derive the Indirect utility fn.

$$x_1^* = \left(\frac{a}{a+b}\right)\left(\frac{M}{P_1}\right) \qquad x_2^* = \left(\frac{b}{a+b}\right)\left(\frac{M}{P_2}\right)$$

$$u^* = x_1^{*a} x_2^{*b} = \left[\left(\frac{a}{a+b}\right)\left(\frac{M}{P_1}\right)\right]^a \left[\left(\frac{b}{a+b}\right)\left(\frac{M}{P_2}\right)\right]^b$$

$$= \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b \left(\frac{M}{P_1}\right)^a \left(\frac{M}{P_2}\right)^b$$

$$= \boxed{\frac{a^a b^b}{(a+b)^{(a+b)}} \cdot \frac{M^{a+b}}{P_1^a P_2^b}} = V(M, P_1, P_2)$$

Q. Given the Indirect utility fn ⇒ Marshallian Demand curves.

(*) Roy's Identity: $x_i^* = -\left(\frac{\partial V/\partial P_i}{\partial V/\partial M}\right)$, $i = 1, 2$.

Given: $V(M, P_1, P_2) = \frac{a^a b^b}{(a+b)^{(a+b)}} \cdot \frac{M^{a+b}}{P_1^a P_2^b}$. Derive the Marshallian demand curves.

For Good 1: $x_1^* = -\left(\frac{\partial V/\partial P_1}{\partial V/\partial M}\right)$

$$\frac{\partial V}{\partial P_1} = \frac{a^a \cancel{b^b}}{(\cancel{a+b})^{(a+b)}} \cdot \frac{M^{a+b}}{\cancel{P_2^b}} (-a) \cdot P_1^{-a-1}$$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{\partial V}{\partial M} = \left(\frac{a^a \cancel{b^b}}{(\cancel{a+b})^{(a+b)}}\right) \frac{(a+b) M^{(a+b-1)}}{P_1^a \cancel{P_2^b}}$$

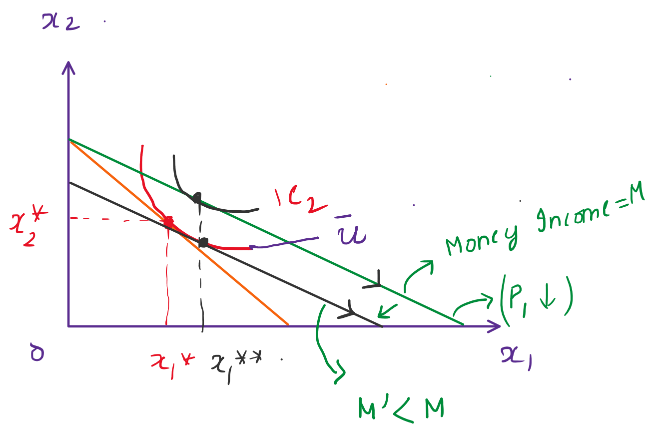
$$x_1^* = -\left(\frac{\partial V/\partial P_1}{\partial V/\partial M}\right) = \cancel{-}\left(\frac{M^{a+b}(\cancel{-}a) P_1^{-a-1}}{\frac{(a+b) \cdot M^{(a+b-1)}}{P_1^a}}\right)$$

$$= \frac{M^{a+b}\, a\, P_1^{-a-1}\, P_1^{a}}{(a+b)\cdot M^{(a+b-1)}}$$

$$= \left(\frac{a}{a+b}\right)\left(\frac{M}{P_1}\right)$$

HW: Solve $x_2^* = -\left(\frac{\partial V/\partial P_2}{\partial V/\partial M}\right)$

Hicksian Demand Curves:



Idea: Fix level of utility and obtain the optimal levels of consumption of the good for various price levels. This gives the hicksian demand curve.

Q. $u = x_1 \cdot x_2$. Find the Hicksian demand curves for the 2 goods.

Fix a level of utility $u = \bar{u}$.

Find the minimum expenditure needed to maintain $u = \bar{u}$.

Minimize$_{\{x_1, x_2\}}$ $P_1 x_1 + P_2 x_2$ subject to $\bar{u} = x_1 x_2$.

$$\mathcal{L} = (P_1 x_1 + P_2 x_2) + \mu[\bar{u} - x_1 x_2]$$ μ: Lagrange multiplier

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \Rightarrow P_1 - \mu x_2 = 0 \Rightarrow P_1 = \mu x_2$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \Rightarrow P_2 - \mu x_1 = 0 \Rightarrow P_2 = \mu x_1$$

$$\Rightarrow \frac{P_1}{P_2} = \frac{x_2}{x_1}$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 0 \Rightarrow \bar{u} - x_1 x_2 = 0 \Rightarrow \bar{u} = x_1 x_2$$

Find x_1^h, x_2^h.

∂x_2

$\frac{\partial \mathcal{L}}{\partial \mu} = 0 \Rightarrow \quad \bar{u} - x_1 x_2 = 0 \quad \Rightarrow \quad \bar{u} = x_1 x_2 \longrightarrow$ Find x_1^h, x_2^h.

$$\frac{P_1}{P_2} = \frac{x_2}{x_1} \Rightarrow x_2 = \left(\frac{P_1}{P_2}\right) x_1$$

Replace $\bar{u} = x_1 \cdot \left(\frac{P_1}{P_2}\right) x_1 = x_1^2 \cdot \frac{P_1}{P_2}$.

$$x_1^2 = \frac{P_2}{P_1} \bar{u} \Rightarrow x_1^h = \sqrt{\frac{P_2}{P_1} \bar{u}} = x_1^h(P_1, P_2, \bar{u})$$

Given $\bar{u}$, x_1^h is the expenditure minimizing level of Good 1.

$$x_2^h = \frac{P_1}{P_2} \cdot x_1^h = \frac{P_1}{P_2} \cdot \sqrt{\frac{P_2}{P_1} \bar{u}} = \sqrt{\frac{P_1}{P_2} \bar{u}} = x_2^h(P_1, P_2, \bar{u})$$

Expenditure $E = P_1 x_1^h + P_2 x_2^h$ [Minimum Expenditure]

$$= P_1 \cdot \sqrt{\frac{P_2}{P_1} \bar{u}} + P_2 \sqrt{\frac{P_1}{P_2} \bar{u}}$$

$$= \sqrt{P_1 P_2 \bar{u}} + \sqrt{P_1 P_2 \bar{u}}$$

$$= 2\sqrt{P_1 P_2 \bar{u}}$$

[Given P_1, P_2 & $\bar{u}$, E represents the minimum amt of exp to be incurred $\Rightarrow$ Expenditure Function].

Expenditure fn given $u = x_1 \cdot x_2$.