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Oscillatory Sequence

A. Finite Sequence

1. Bounded limit
2. It will neither converge, nor diverge.

B. Infinite Oscillatory Sequence

1. Unbound (not bounded)
2. same as above

Ex: $\{1+(-1)^n\} = \{0, 2, 0, 2, 0, 2, \dots\}$

$\{(-1)^n \cdot n\} = \{-1, 2, -3, 4, -5, 6, \dots\}$

Bernoulli's Inequality:

For every positive integer $n \geq 2$ and $1+p > 0$,

$$\boxed{(1+p)^n > 1+np}$$

Proof: when $n=2$

$$(1+p)^2 = 1+2p+p^2 > 1+2p$$

$\therefore$ inequality will hold for $n=2$

Let us assume that the inequality holds for any particular value k of n

$(1+p)^k > 1+kp$

we will assume

→ i.e. we assume $(1+p)^k > 1+kp$

Let us multiply both sides by $(1+p)>0$

$$(1+p)^k(1+p) > (1+kp)(1+p)$$

$$(1+p)^{k+1} > (1+kp)+(1+kp)p$$

$$1+kp+p+kp^2$$

$$(1+p)^{k+1} > \underbrace{p(1+k)}_{n}+kp^2$$

$$> (1+k)p$$

Thus it is observed that the inequality holds for $n=k+1$ if it holds for $n=k$.

It has been proved that the inequality is true for $n=2$, so it is true for $n=2+1=3$

By same logic if it is true for $n=3$ it will hold for $n=4$ as well and so on.

Thus the inequality is true for any integral value of $n \geq 2$.

(Proved)

<u>Null Sequence</u>: A sequence is said to be a null

Null Sequence: A sequence is said to be a null sequence, if

$$\lim_{n\to\infty} x_n = 0$$

ie for any +ve no. ε, however small, there exist a positive integer N, such that $|x^n| < \varepsilon$ for all $n > N$.

Theorem: The sequence $\{x^n\}$ is a null sequence if $|x| < 1$.

$\frac{1}{h\varepsilon}$

We note that if $x = 0$. each term of sequence is 0 and so $x^n \to 0$ as $n \to \infty$

when, $|x|$ be positive proper fraction.

so we take $|x| = \frac{1}{1+h}$ where $h > 0$.

Now $|x^n| = \frac{1}{(1+h)^n} < \frac{1}{1+nh}$

(by Bernoulli's inequality)

$\therefore |x^n| < \frac{1}{nh} < \varepsilon$, if $n > \frac{1}{h\varepsilon}$

or $|x^n - 0| < \varepsilon$ if $n \geq \left[\frac{1}{h\varepsilon}\right] + 1 = N$ (say)

$[h_\varepsilon]$ (say)

Hence, the sequence $\{x^n\}$ is a null sequence if $|x| < 1$.

Theorems on Sequence & Monotone Sequence.

1. A convergent sequence determines its limit uniquely.

2. Every convergent sequence is bounded.

3. If $L \le x_n \le M$ for all n and if $\lim_{n\to\infty} x_n$ exist and $\lim_{n\to\infty} x_n = l$ then $L \le l \le M$.

Corollary: If $\{x_n\}$ is a sequence of positive terms then $\lim_{n\to\infty} x_n \geq 0$ provide that the limit exists.

4. A monotonic increasing sequence, which is bounded above is convergent and converges to its exact upper bound or supremum.

5. A monotonic decreasing sequence, which is bounded below is convergent and converges to its exact lower bound.

(6) A monotonic increasing sequence diverges to $+\infty$, if it is not bounded above.

(7) A monotonic decreasing sequence diverges to $-\infty$, if not bounded below.

(8) Squeeze Theorem:

Suppose $\{x_n\}$ $\{y_n\}$ and $\{z_n\}$ are sequences of real no's such that for some integer m, $x_n \le y_n \le z_n$ for all integer $n \geq m$.

and that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$

then the sequence $\{y_n\}$ is convergent and $\lim_{n\to\infty} y_n = l$ also.

Some Applications:

(1) Find the limit of the sequence $\{\frac{1}{n}\}$ as $n \to \infty$.

we observe that here $\left|\frac{1}{n} - 0\right| < \varepsilon$, for $n \not< \frac{1}{\varepsilon}$

$n < N$

N

Taking $N = \frac{1}{\varepsilon}$ or the integral part of ε,

when ε is a fraction we may write

$$\left|\frac{1}{n} - 0\right| < \varepsilon, \text{ if } n \geq N$$

$$\therefore \lim_{n\to\infty} \frac{1}{n} \to 0$$

$\therefore \lim_{n\to\infty} \frac{1}{n} \to 0$

(Proved).

(2) Show that $\lim_{n\to\infty} x_n = 1$, where $x_n = 1 + \frac{(-1)^2}{n}$

if $\varepsilon > 0$ be any no., $|x_n - 1| = \left|1 + \frac{(-1)^2}{n} - 1\right| = \frac{1}{n} < \varepsilon$

if $n > 1/\varepsilon$

if M be a positive integer $> \frac{1}{\varepsilon}$ then $|x_n - 1| < \varepsilon$

for all $n \geq M$

then $\lim_{n\to\infty} x_n = 1$. (Proved).

(3) Show the sequence $\{x_n\}$, where $x_n = 2 - \frac{1}{2^n}$, is convergent.

Note: $|x_n - l| < \varepsilon$ for $n \geq N$

i.e. $l - \varepsilon < x_n < l + \varepsilon$ when $n \geq N$

$\{x_n\}$ converges to l i.e. $\lim_{n\to\infty} x_n = l$

Here $|x_n - 2| = \left|-\frac{1}{2^n}\right| = \frac{1}{2^n}$

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Let, $\epsilon > 0$ be given, then $|x_n - 2| < \epsilon$, if $\frac{1}{2^n} < \epsilon$

ie $2^n > \frac{1}{\epsilon}$

or, $n \log 2 > \log(1/\epsilon)$

or, $n > \frac{\log(1/\epsilon)}{\log 2}$

$$n \geq M \geq \frac{\log 1/\epsilon}{\log 2}$$

if M be a positive integer $\geq \frac{\log(1/\epsilon)}{\log 2}$

then $|x_n - 2| < \epsilon$; for $n \geq M$

$\lim_{n \to \infty} x_n = 2$. (Proved)