Q. Given the system:

$$x_1 + x_3 = 0$$
$$x_1 + 2x_2 + 3x_3 = 0$$
$$-x_1 + x_2 = 0$$

Find the nullity of the system. [Dimension of the solution space]

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix}_{3\times 3}$$

unique soln $\Rightarrow |A| \neq 0$.

Infinite soln $\Rightarrow |A| = 0$.

$$N(A) = \{ \underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0} \}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in N(A) \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$|A| = 0 \Rightarrow$ Infinite solution.

(i): $x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 = -k$.

(iii): $-x_1 + x_2 = 0 \Rightarrow x_1 = x_2 = k$.

$\therefore \underset{\sim}{x} \in N(A) \Rightarrow \begin{bmatrix} k \\ k \\ -k \end{bmatrix} \in N(A) \Rightarrow k\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in N(A)$

$$N(A) = \left\{ k\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, k \in \mathbb{R} \right\}$$

Nullity [Dim of Null space]

Basis of $N(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \Rightarrow$ Nullity $= 1$.

Q. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation s.t $T(x, y) = (x+y, x-y, y)$. Find the dimension of its range space.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2' \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3' \to 2R_3 + R_2} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

↳ 2 non-zero rows

Dim $R(T) = 2$.

(*) A basis of the R(T):

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(*) A basis of the $R(T)$:

$\mathbb{R}^2$: basis of $\mathbb{R}^2 = \{(1,0), (0,1)\}$ ⇒ standard basis of $\mathbb{R}^2$

$T(1,0)$ and $T(0,1)$ $\in \mathbb{R}^3$. [Range space].

$T(1,0) = (1,1,0)$

$T(0,1) = (1,-1,1)$

$T(x,y) = (x+y, x-y, y)$

$T(1,0) = (1+0, 1-0, 0) = (1,1,0)$

$\therefore \{(1,1,0), (1,-1,1)\}$ are l.i ?

↳ Qualify to be a basis of $R(T)$

$c_1(1,1,0) + c_2(1,-1,1) = (0,0,0)$.

$c_1 = 0, \quad c_2 = 0 \Rightarrow \{(1,1,0),(1,-1,1)\}$ are l.i

↳ Basis of $R(T)$.

Q. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x_1,x_2,x_3) = (x_1-x_2, x_1-x_3, 0)$

Then: (a) dim $N(T) = 2$ (X) ~~(b)~~ dim $R(T) = 2$

(c) $R(T) = N(T)$ (X) (d) $N(T) \subset R(T)$ (X)

Let $(x_1,x_2,x_3) \in N(T)$.

$T(x_1,x_2,x_3) = (0,0,0)$

$(x_1-x_2, x_1-x_3, 0) = (0,0,0)$

$$\left.\begin{array}{l} x_1 - x_2 = 0 \\ x_1 - x_3 = 0 \end{array}\right\} \Rightarrow x_1 = x_2 = x_3 = k \text{ (say)}.$$

$\therefore (k,k,k) \in N(T)$.

$N(T) = \{k(1,1,1), k \in \mathbb{R}\}$.

dim $N(T) = 1$ [Nullity].

Rank-Nullity Th: Rank [dim $R(T)$] = dim of V − Nullity

$= 3 - 1 = 2$.

$$N(T) = \{ k(1,1,1),\ k \in \mathbb{R} \}.$$

$$= \{ (1,1,1),\ (2,2,2),\ (0,0,0) \}$$

Q. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ $T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3,\ 3x_1 + 4x_2 + x_3,\ 2x_1 + x_2 - x_3)$

Dim of range space of T^2 is:

(a) 0 (b) 1 (c) 2 (d) 3.

T^2 = Product transformation = $T \cdot T$

$$T = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 14 & 17 & 3 \\ 17 & 26 & 9 \\ 3 & 9 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2' \to R_2 - (R_1 + R_3)} \begin{bmatrix} 14 & 17 & 3 \\ 0 & 0 & 0 \\ 3 & 9 & 6 \end{bmatrix} \Rightarrow \text{dim of range space} = 2 \text{ (c)}$$