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Lagrangian Multiplier

Cobb Douglas : $Q = A \cdot K^{\alpha} L^{\beta}$

Q = Output

A = technology / innovation etc (constant)

K = capital

L = Labour

α = share of Capital in production

β = Share of Labour in production.

$K, L > 0$

$Q > 0$

$$\left\{ \begin{array}{c} A > 0 \\ 0 < \alpha, \beta < 1 \end{array} \right\}$$

α = output elasticity of capital (percentage change in output due to percentage change in capital)

$$\alpha = \frac{\frac{\partial Q}{Q} \times 100}{\frac{\partial K}{K} \times 100} = \frac{\partial Q}{\partial K} \cdot \frac{K}{Q}$$

Similarly $\beta \rightarrow$ w.r.t Labour (L)

Constraint Optimisation of Cobb-Doughlas production fn

↓

(Lagrange)

$0 < \alpha, \beta < 1$

(*) Objective is to maximise production level (Q) with given cost of production (budget).

Q Cobb Douglas produ fn $Q = K^{0.4} L^{0.5}$

$P = 3$ $\quad$ $P = 4$

Q Cobb Doug… p…

$C = \$108 \qquad P_K = 3 \qquad P_L = 4$

Cost fn : $\boxed{C = P_K \cdot K + P_L \cdot L}$

$\boxed{108 = 3\underline{K} + 4\underline{L}}$

Obj. max $\boxed{Q = K^{0.4} L^{0.5}}$

s.t. $\underline{108} = 3K + 4L$ (constraint)

The Lagrangian expression can be written as

$$Z = \left(\underline{K}^{0.4}\, \underline{L}^{0.5}\right) + \underline{\lambda}\left(108 - 3K - 4L\right)$$

F.O.C

$$\frac{\partial Z}{\partial K} = 0.4K^{-0.6} L^{0.5} - 3\lambda = 0$$

$$\lambda = \frac{0.4K^{-0.6} L^{0.5}}{3} \quad \text{--- (1)}$$

$$\frac{\partial Z}{\partial L} = 0.5L^{-0.5} K^{0.4} - 4\lambda = 0$$

or, $\lambda = \dfrac{0.5L^{-0.5} K^{0.4}}{4}$ --- (2)

$$\frac{\partial Z}{\partial \lambda} = 108 - 3K - 4L = 0$$

$\Rightarrow 3K + 4L = 108$ --- (3)

$\Rightarrow 3K+4L=108$ — ②

Comparing ① and ②

$$\frac{0.4K^{-0.6}L^{0.5}}{3} = \frac{0.5L^{-0.5}K^{0.4}}{4}$$

or, $\frac{0.4L^{0.5}}{3L^{-0.5}} = \frac{0.5K^{0.4}}{4K^{-0.6}}$

or, $0.4\frac{L}{3} = 0.5\frac{K}{4}$

or, $\frac{0.4}{0.5}\times\frac{4}{3}\times L = K$

or, $\frac{1.6}{1.5}L = K$

or, $\boxed{\frac{16}{15}L = K}$ — ④

Substitute the value of K in eq ③,

$$108 = 3K + 4L$$

or, $108 = \cancel{3}\times\frac{16}{\cancel{15}_5}L + 4L$

or, $108\times 5 = 16L + 20L$

or, $108\times 5 = 36L$

$L = \frac{\cancel{108}^{3}\times 5}{\cancel{36}}$ $= 15$ units

$108 = \cancel{3}\times\frac{16}{\cancel{15}_5}L + 4L$

$108\times 5 = 16L + 20L$

$108\times 5 = 36L$

$L = \frac{3\ \cancel{108}\times 5}{\cancel{36}}$

$L = 15$ units

$L = \frac{\cancel{408} \times 3}{\cancel{86}\ 8} = 15$ units

$\therefore\ K = \frac{16}{15} \times L = \frac{16}{\cancel{15}} \times \cancel{15} = 16$ units

max $Q = K^{0.4} L^{0.5} = 16^{0.4}\ 15^{0.5}$

$=$ ans

elasticity of ~~labour~~ output w.r.t labour

$L = 15$

$K = 16$

$Q =$

$$e_{Q,L} = \frac{\partial Q}{\partial L} \cdot \frac{L}{Q}$$

$$= \cancel{K^{0.4}} \cdot 0.5 \cdot L^{-0.5} \times \frac{15}{\cancel{K^{0.4}} \cdot L^{0.5}}$$

$$= 0.5 \cdot \frac{15}{L}$$

$$= 0.5 \times \frac{\cancel{15}}{\cancel{15}}$$

$= 0.5$

$= \beta$

$Q = L^1 \cdot K^1 =$

$$e_{Q,L} = \frac{\partial Q}{\partial L} \cdot \frac{L}{Q} = K \cdot \frac{L}{Q} = K \frac{L}{LK} = 1$$

$$e_{Q,K} = \frac{\partial Q}{\partial K} \cdot \frac{K}{Q}$$

$$e_{Q,K} = \frac{\partial Q}{\partial K} \cdot \frac{K}{Q} = L \cdot \frac{K}{Q} = \cancel{L} \cdot \frac{\cancel{K}}{\cancel{L} \cdot \cancel{K}} = 1$$

1. Homogeneous degree of

2. Lagrange for (optimisation with constraint equality)

3. slope and curvature

4. $\boxed{\text{Elasticity of substitution}}$