16 March 2024 10:06 PM

Limits

Indeterminate forms: $\frac{\infty}{\infty}$ $\frac{0}{0}$ 1^{∞} ∞^{0} $\infty \times 0$ . . .

L'Hospital's Rule:

if $\lim_{x \to a} \frac{f(x)}{g(x)}$ reduces to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then differentiate numerator & denominator till this form is removed.

i.e. $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

If we get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again, Then we will differentiate it again

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$$

Ex 1 Evaluate $\lim_{x \to 2} \frac{x^6 - 24x - 16}{x^3 + 2x - 12}$

Sol^n: $\lim_{x \to 2} \frac{x^6 - 24x - 16}{x^3 + 2x - 12}$ $\left[\frac{0}{0} \text{ form}\right]$

$\hookrightarrow$ by L'Hospital's rule

$$\lim_{x \to 2} \frac{6x^5 - 24}{3x^2 + 2}$$

$$\frac{6(2)^5 - 24}{3(2)^2 + 2} = \frac{168}{14} = 12$$

Frequently used series Expansions:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$a^x = 1 + \frac{x \cdot \log a}{1!} + \frac{(\log a)^2 x^2}{2!} + \cdots$$

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}$$

$$(1+x)^{1/x} = e\left[1 - \frac{x}{2} + \frac{11x^2}{24} + \cdots\right]$$

$$\sin x = x - \frac{x^3}{} + x^5 - x^7 \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{316}x^7 + \cdots$$

$$\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$$

$$(\sin^{-1} x)^2 = \frac{2}{2!}x^2 + \frac{2 \cdot 2^2}{4!}x^4 + \frac{2 \cdot 2^2 \cdot 4^2}{6!}x^6 + \cdots$$

$$x \cot x = 1 - \frac{x^3}{3} + \frac{x^4}{45} + \frac{2x^6}{945} + \cdots$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

$$x \operatorname{cosec} x = 1 + x^2 \quad 7x^4 \quad 31x^6 + \cdots$$

$$x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \cdots$$

① Evaluate $\displaystyle\lim_{x\to 0} \frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$

$$\lim_{x\to 0} \frac{x\left[2x + \frac{2^3x^3}{3} + \frac{2\,2^5x^5}{15} + \cdots\right] - 2x\left[x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right]}{(2\sin^2 x)^2}$$

$$\lim_{x\to 0} \frac{x^4\left[\frac{8}{3} - \frac{2}{3}\right] + x^6\left[\frac{64}{15} - \frac{4}{15}\right] + \cdots}{4\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^4}$$

$$\lim_{x\to 0} \frac{2 + 4x^2 + \cdots}{4\left(1 - \frac{x^2}{3!} + \cdots\right)^4} = \frac{2}{4} = \frac{1}{2} \text{ (ans)}.$$

② Evaluate $\displaystyle\lim_{x\to 1} \frac{x^3 - x^2\log x + \log x - 1}{x^2 - 1}$

$$\lim_{x\to 1} \frac{(x^3-1) - (x^2-1)\log x}{x^2-1} = 3/2$$

$$\lim_{x\to 1} \frac{\ldots}{x^2-1} = 3/2$$ (ans)

(3) Evaluate $\lim\limits_{x\to 2a} \dfrac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}}$

$$= \lim_{x\to 2a} \frac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{(x-2a)(x+2a)}}$$

$$= \lim_{x\to 2a} \frac{\sqrt{x-2a} + \dfrac{x-2a}{\sqrt{x}+\sqrt{2a}}}{\sqrt{x-2a}\,\sqrt{x+2a}}$$

$$= \lim_{x\to 2a} \frac{1}{\sqrt{x+2a}} + \frac{\sqrt{x-2a}}{\sqrt{x+2a}\,(\sqrt{x}+\sqrt{2a})}$$

$$= \frac{1}{\sqrt{4a}} = \frac{1}{2\sqrt{a}}$$

(4) $\lim\limits_{x\to 2} \dfrac{x^3-2^3}{x-2} = 12$ (ans)

(5) Evaluate $\lim\limits_{x\to 1} \dfrac{x+x^2+\cdots+x^n-n}{}$

$x - 1$

$\boxed{\dfrac{n(n+1)}{2}}$ ans.

(6) Evaluate $\lim_{n \to \infty} \dfrac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$

$$\lim_{n \to \infty} \frac{(n+2)(n+1)! + (n+1)!}{(n+2)(n+1)! - (n+1)!}$$

$$\frac{\cancel{(n+1)!}\,[n+2+1)}{\cancel{(n+1)!}\,[n+2-1)}$$

$$\lim_{n \to \infty} \frac{n+3}{n+1} = \lim_{n \to \infty} \frac{1+3/n}{1+1/n} = 1 \text{ (ans)}$$

(7) Evaluate $\lim_{x \to \infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right)$

$$= \lim_{x \to \infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right) \times \frac{\left(\sqrt{x^2+x+1} + \sqrt{x^2+1}\right)}{\sqrt{x^2+x+1} + \sqrt{x^2+1}}$$

$$\lim_{x \to \infty} \frac{(x^2+x+1) - (x^2+1)}{\sqrt{x^2+x+1} + \sqrt{x^2+1}}$$

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2+x+1} + \sqrt{x^2+1}}$$

$$\lim_{x \to \infty} \frac{\quad}{\sqrt{x^2+x+1} + \sqrt{x^2+1}}$$

$$= \frac{1}{2} \quad (\text{ans}).$$

(8) If $\lim\limits_{x \to \infty} \left(\dfrac{x^2+1}{x+1} - ax - b \right) = 0$

find the values of a and b.

Given, $\lim\limits_{x \to \infty} \left(\dfrac{x^2+1}{x+1} - ax - b \right) = 0$

$$\lim_{x \to \infty} \frac{x^2+1 - ax^2 - ax - bx - b}{x+1} = 0$$

$$\lim_{x \to \infty} \frac{x^2(1-a) - x(a+b) + (1-b)}{x+1} = 0$$

deg of num < deg of denominator

$1 - a = 0$ and $a + b = 0$

$a = 1$ $\qquad$ $b = -1$

(9) If $\lim\limits_{x \to \infty} \left(\dfrac{x^2+1}{x+1} - ax - b \right) = 2$

$\lim_{x\to\infty} \left(x+1 \right) =$

find the values of a and b.

(10) If $\lim_{x\to\infty} \dfrac{x^2+1}{x+1} - ax - b = \infty$

find 'a' and 'b'.

(11) Evaluate $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

(12) $\lim_{x\to 0} \dfrac{1-\cos(1-\cos x)}{\sin^4 x}$

$\lim_{x\to 0} \dfrac{1-\cos(1-\cos x)/x^4}{\sin^4 x/x^4}$

$$\lim_{x\to 0} \frac{\dfrac{1-\cos\left(2\sin^2 x/2\right)}{x^4}}{\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^4}$$

$\lim_{x\to 0} 2\left[\dfrac{\sin\left(\sin^2\frac{x}{2}\right)}{\sin^2 x} \quad \cancel{\sin^2}\dfrac{x}{2}\right]^2$

$$\lim_{x\to 0} 2\left[\frac{\ldots}{\cancel{\sin^2 x/2} \quad \cancel{4}\cdot\frac{x^2}{\cancel{4}}}\right]$$

$$= 2 \times \frac{1}{4^2} = \frac{1}{8} \text{ (ans)}$$