# Solutions

(1) The value of $\lim_{n\to\infty} \frac{[x]+[2^2x]+[3^2x]+\cdots+[n^2x]}{n^3}$ (where $[\cdot]$ represents the greatest integer function) is

(A) $x$ (B) $\frac{x}{2}$ (C) $\frac{x}{3}$ (D) $\frac{2x}{3}$ (E) none of these

$$\lim_{n\to\infty} \frac{\sum (n^2x-1)}{n^3} \qquad \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} \frac{\sum n^2x}{n^3}$$

$$\left.\begin{aligned} x-1 &< [x] \le x \\ 2^2x-1 &< [2^2x] \le 2^2x \\ 3^2x-1 &< [3^2x] \le 3^2x \\ n^2x-1 &< [n^2x] \le n^2x \end{aligned}\right\} \text{By definition of the greatest integer function}$$

(2) The value of $\lim_{n\to\infty} n^{-n^2}\left((n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\cdots\left(n+\frac{1}{2^{n-1}}\right)\right)^n =$

(A) 1 (B) $e$ (C) $e^2$ (D) none of these

$$L = \lim_{n\to\infty}\left(\frac{(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\cdots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right)^n\left(1+\frac{1}{2^2n}\right)^n\cdots\left(1+\frac{1}{2^{n-1}n}\right)^n$$

(use standard form)

$$= e\cdot e^{\frac{1}{2}}\cdot e^{\frac{1}{2^2}}\cdots e^{\frac{1}{2^{n-1}}}\cdots\infty = e^2 \text{ (how?)}$$

(use infinite GP sum)

(3) The value of $\lim_{x\to\frac{\pi}{2}} \frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ln(\sin x)}$ is

(A) 1 (B) 2 (C) 3 (D) 4

$\sin x = t$, $t = 1+h$

$$L = \lim_{t\to1}\frac{t-t^t}{1-t+\ln t} = \lim_{h\to0}\frac{(1+h)-(1+h)^{1+h}}{-h+\ln(1+h)} = \lim_{h\to0}\frac{(1+h)\{1-(1+h)^h\}}{-h+\{h-\frac{h^2}{2}+\frac{h^3}{3}-\cdots\}} = \lim_{h\to0}\frac{(1+h)\{1-(1+h^2+h\frac{(h-1)h^2}{2!}+\cdots)\}}{-\frac{h^2}{2}+\frac{h^3}{3}h-+-\cdots} = 2$$

$$\frac{-h^2}{-h^2/2} \qquad \frac{-1}{-1/2}$$

(4) The value of $\lim_{x\to0^+}\left(\frac{1}{x}\right)^{\sin x}$

(A) 0 (B) 1 (C) $e$ (D) None of these

This limit is of the indeterminate form $\infty^0$. Lets first convert it into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking its logarithm:

$$\ln L = \lim_{x\to0^+}\ln\left(\frac{1}{x}\right)^{\sin x} = \lim_{x\to0^+}\sin x\cdot\ln\left(\frac{1}{x}\right) = -\lim_{x\to0^+}\sin x\cdot\ln x$$

$$= -\lim_{x\to0^+}\frac{\ln x}{\operatorname{cosec} x} = -\lim_{x\to0^+}\frac{1/x}{-\operatorname{cosec}\cot x} \text{ (By applying the LH rule)}$$

$$= \lim_{x\to0^+}\frac{\sin^2 x}{x\cos x} = \lim_{x\to0^+}\left(\frac{\sin x}{x}\right)\cdot\tan x = 0$$

# Algebra

$a \neq 0$

$$f(x) = ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

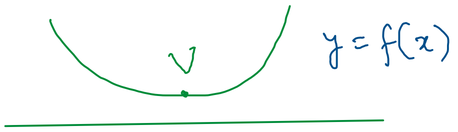
$$\Rightarrow \quad y + \frac{D}{4a} = a\left(x+\frac{b}{2a}\right)^2$$

$(x-\alpha)^2 = 4k(y-\beta)$ vertex $(\alpha, \beta)$

Comparing → vertex of quadratic parabola $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

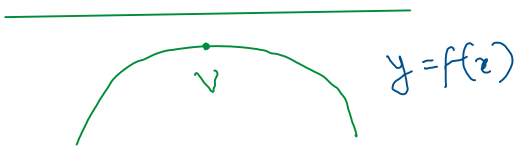
(1) $a > 0$, $D < 0$

$V\left(\frac{-b}{2a}, \frac{D}{4a}\right)$



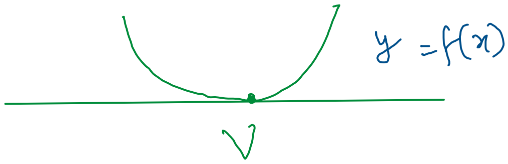

(2) $a < 0$, $D < 0$

$V\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$




(3) $a > 0$, $D = 0$

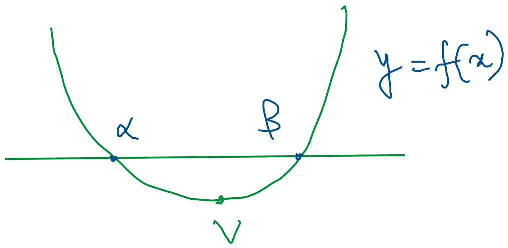
$V\left(\frac{-b}{2a}, 0\right)$




(4) $a < 0$, $D = 0$

(5) $a > 0$, $D > 0$

$V\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$




(6) $a < 0$, $D > 0$

**Illustration 6.18** If $a, b, c \in R$, $a \neq 0$ and the quadratic equation $ax^2 + bx + c = 0$ has no real root, then show that $(a + b + c)\, c > 0$.

$f(x) = ax^2 + bx + c = 0$

$f(0) \cdot f(1) > 0$

$\Rightarrow \quad c(a+b+c) > 0$