pmf (x discrete random variable)

1. $f(x) \geq 0 \quad \forall x$

2. $\sum_x f(x) = 1$.

pdf (x cont - random variable)

1. $f(x) \geq 0 \quad \forall x$

2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

1. Binomial Distribution (pmf of a discrete r.v)

a. 'n' repeated trials -

b. probab of success $= p$

c. " " failure $= q$

d. $p+q=1$

Let x be the no. of success in 'n' trials where x can assume any value $0, 1, 2 \cdots, n$.

If $x =$ no. of succes

$n-x =$ no. of failure.

$\therefore$ probability of x successes is given by

$$f(x) = {}^nC_x\, p^x\, q^{n-x}. \quad \text{for } x = 0, 1, \cdots, n$$

$$= 0 \quad \text{otherwise}$$

This is the p.m.f of Binomial Distribution with parameter n and p.

① (a) $f(x) = {}^nC_x\, p^x\, q^{n-x}$

$\underline{(a+b)^n = {}^nC_0 \cdots\cdots {}^nC_n}$

(1) $f(x) = {}^nC_x p^x q^{n-x}$

(a) Here, ${}^nC_x > 0$ $p^x > 0$ $q^{n-x} > 0$

$\therefore f(x) > 0$ for all x.

(b) $\sum_x f(x) = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}$

$= {}^nC_0 p^0 q^{n-0} + {}^nC_1 p^1 q^{n-1} + \cdots + {}^nC_n p^n q^0$

$= (p+q)^n$

$\therefore f(x)$ is a pmf. $= (1)^n = 1$ (proved)

Mean of Binomial Distribution

$E(x) = \sum_x x f(x) = \sum_{x=0}^{n} x \, {}^nC_x p^x q^{n-x}$

$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$

$= \sum_{x=0}^{n} \cancel{x} \cdot \frac{n!}{(n-x)!\,\cancel{x}(x-1)!} p^x q^{n-x}$

$= \sum_{x=1}^{n} \frac{n(n-1)!}{(n-x)!\,(x-1)!} p^x q^{n-x}$

$= np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!(x-1)!} p^{x-1} q^{n-x}$

$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1} p^{x-1} q^{n-x}$

$= np \, (p+q)^{n-1}$

${}^nC_x = \frac{n!}{x!(n-x)!}$

${}^{n-1}C_{x-1} = \frac{(n-1)!}{(x-1)!(n-x)!}$

$x \; a^{n-x}$

$$\sum {}^{n-1}C_x\, p^x q^{n-x}$$

$$(p+q)^{n-1}$$

$$= np\,(p+q)^{n-1}$$

$$= np \cdot 1$$

$$= np = E(x) \rightarrow$$ mean of binomial distribution.

(2)

Variance of Binomial Distribution:

$$V(x) = E(x^2) - [E(x)]^2$$

$$V(x) = E(x^2) - (np)^2$$

$$= E[x(x-1)+x] - (np)^2$$

$$= E(x(x-1)) + E(x) - (np)^2$$

$$\boxed{V(x) = E(x(x-1)) + np - (np)^2}$$

Now $$E[x(x-1)] = \sum_{x=0}^{n} x(x-1) f(x)$$

$$= \sum_{x=2}^{n} x(x-1)\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1) \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \cancel{x}(\cancel{x-1}) \frac{n(n-1)(n-2)!}{(n-x)!\,\cancel{x}(\cancel{x-1})(x-2)!} p^x q^{n-x}$$

$$= n(n-1)p^2 \left(\sum^{n} \frac{(n-2)!}{\quad} p^{x-2} q^{n-x}\right.$$

$$= n(n-1)p^2 \left(\sum_{x=2}^{n} \frac{(n-2)!}{(n-x)!\,(x-2)!} p \quad q \right)$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-x}$$

$$= n(n-1)p^2 \,(p+q)^{n-2}$$

$$E[x(x-1)] = n(n-1)p^2$$

$$\therefore V(x) = n(n-1)p^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$\boxed{\begin{aligned} V(x) &= npq \\ S.D(x) &= \sqrt{npq} \end{aligned}} \rightarrow \text{var of B.S.}$$

$(n+1)p$ is an integer

$(n+1)p$ non integer (fractional) $\rightarrow$ unimodal

Moment generating function (Binomial Distribution).

Moment generating function (Binomial Distribution).

$$M(t) = E(e^{tx}) = \sum_x e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x\, (pe^t)^x q^{n-x}$$

$$= {}^nC_0 (pe^t)^0 q^n + {}^nC_1 (pe^t)^1 q^{n-1} + \dots + {}^nC_n (pe^t)^n q^0$$

$$\boxed{M(t) = (pe^t+q)^n}$$

$$\left.\frac{d\,M(t)}{dt}\right|_{t=0} = n(pe^t+q)^{n-1} \cdot pe^t$$

$$= n(pe^0+q)^{n-1} \cdot pe^0$$

$$= np = \text{mean}$$

$$= \mu_1'$$

$$\left.\frac{d^2 M(t)}{dt^2}\right|_{t=0} = np\left[(pe^t+q)^{n-1} e^t + e^t (n-1)(pe^t+q)^{n-2} \cdot pe^t\right]$$

$$= npe^t\left[(pe^t+q)^{n-1} + (n-1)pe^t(pe^t+q)^{n-2}\right]$$

At $t=0$,

$$= npe^0\left[(pe^0+q)^{n-1} + (n-1)pe^0(pe^0+q)^{n-2}\right]$$

$$= npe^0 \left[(pe + q) + (n-1)pe(pe+q) \right]$$

$$= np\left[1 + (n-1)p\right]$$

$$= n(n-1)p^2 + np = \mu_2'$$

$$\therefore \text{Variance } (\mu_2) = \mu_2' - (\mu_1')^2$$

$$= np + n(n-1)p^2 - (np)^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{(np)^2}$$

$$= np - np^2 = np(1-p) = npq$$

$$\therefore \boxed{s.d = \sqrt{V(x)} = \sqrt{npq}}$$

Poisson Approximation of BD

as a limiting form of BD

$p \to 0$

i.e $n \to \infty$

then $np = \lambda$

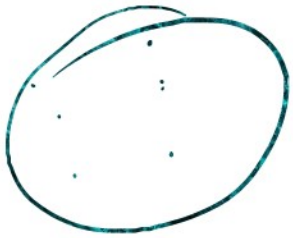
which is finite no.

pmf of BD $\quad f(x) = {}^nC_x \, p^x q^{n-x}$ for $x = 0, 1, \ldots n$.

$$\therefore f(x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

$$\therefore f(x) = \frac{n!}{(n-x)!\,x!}\, p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots\cancel{(n-x)!}}{\cancel{(n-x)!}\;x!}\, p^x q^{n-x}$$



$$= \frac{n(n-1)(n-2)\cdots(n-x+1)\cdot p^x q^{n-x}}{x!}$$

$$= \frac{n}{n}\,\frac{(n-1)}{n}\,\frac{(n-2)}{n}\cdots\frac{(n-x+1)}{n}\cdot\frac{n^x p^x q^{n-x}}{x!}$$

$$f(x) = 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)\frac{(np)^x\, q^{n-x}}{x!}$$

Lt $n\to\infty$, $p\to 0$

$$f(x) = \frac{1\cdot\lambda^x e^{-\lambda}}{x!}$$

$$\therefore \boxed{f(x) = \frac{\lambda^x e^{-\lambda}}{x!}} \quad \text{for } x = 0, 1, 2, \ldots \infty$$

is p.m.f of Poisson Distribn

1. pmf ✓
2. $E(x)$ $V(x)$ ✓
3. Moment ✓

Ref → NG Das