Differential Equations

Given: $y = f(x) \Rightarrow \frac{dy}{dx}$ = Derivative of the function
[Rules of differentiation]

Now: $\frac{dy}{dx} = \phi(x) \Rightarrow$ Find the underlying fn.
[Idea of differential equation]

Eg: $\frac{dy}{dx} = (2x+3)$ Solve this differential equation.

$$\Rightarrow dy = (2x+3)\,dx.$$

Integrating both sides: $\int dy = \int (2x+3)\,dx$

$$\Rightarrow y + c_1 = x^2 + 3x + c_2.$$

$$\Rightarrow y = x^2 + 3x + \underbrace{(c_2 - c_1)}_{c}$$

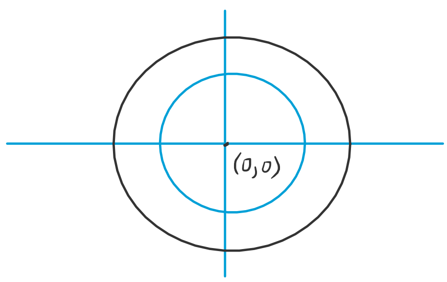
$$\Rightarrow y = x^2 + 3x + c$$

[c = constant of integration].

Q. $\frac{d^2y}{dx^2} + 2\cdot\frac{dy}{dx} + y = 0 \Rightarrow$ Also differential eqn.

Q. $x^2 + y^2 = r^2,\ r > 0$ ..... Family of circles with centre (0,0).

using derivatives,
we can represent the family of circles as a differential Equation.



Order and Degree of Differential Equation:-

Note: $\frac{d^ny}{dx^n}$ = n^{th} order derivative of $y = f(x) = f^{(n)}(x)$.

Note: $\frac{d^n y}{dx^n}$ = n^{th} order derivative of $y = f(x) = f^{(n)}(x)$.

Order = highest order derivative present in the differential equation.

Degree = power attached to the highest order derivative.

Eg: $\frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + y = 10$ .... order = 2. degree = 1.

Eg: $y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ ..... order = 1. .... degree = 2.

$$\underbrace{\left(y - x\cdot\frac{dy}{dx}\right)^2} = a^2\left(1+\left(\frac{dy}{dx}\right)^2\right).$$

Eg: $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = 5\cdot\frac{d^2y}{dx^2}$ .... order = 2. .... degree = 2.

squaring: $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 25\left(\frac{d^2y}{dx^2}\right)^2$.

Eg: $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ ... order = 3. ... degree = 2.

Q. Form a differential equation for the fn: $y = \sin(\sin x)$.

$$y = \sin(\sin x)$$

$$\frac{dy}{dx} = \boxed{\underbrace{\cos(\sin x)}\cdot\cos x} \Rightarrow \frac{1}{\cos x}\cdot\frac{dy}{dx} = \cos(\sin x)$$

$$\frac{d^2y}{dx^2} = \cos x\left[-\sin(\sin x)\cos x\right] + \cos(\sin x)(-\sin x)$$

$$\frac{d^2y}{dx^2} = -\overbrace{\sin(\sin x)}\cdot\cos^2 x - \underbrace{\cos(\sin x)}\cdot\sin x.$$

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \frac{1}{\cos x}\cdot\frac{dy}{dx}\cdot\sin x.$$

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \frac{1}{\cos x} \cdot \frac{dy}{dx} \cdot \sin x$$

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \tan x \cdot \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} + \tan x \cdot \frac{dy}{dx} + y\cos^2 x = 0$$

First order Differential Equations:-

(I) Variable separation Method:-

Q. Solve: $(x^2 - yx^2)\,dy + (y^2 + xy^2)\,dx = 0$

$$(y^2 + x \cdot y^2)\,dx = -(x^2 - yx^2) \cdot dy$$

$$y^2(1+x) \cdot dx = (yx^2 - x^2)\,dy$$

$$y^2(1+x) \cdot dx = x^2(y-1) \cdot dy$$

$$\frac{(1+x)}{x^2}\,dx = \frac{(y-1)}{y^2}\,dy$$

Integrating: $\int \frac{1+x}{x^2}\,dx = \int \frac{y-1}{y^2} \cdot dy$ General soln.

$$\Rightarrow -\frac{1}{x} + \log|x| = \log|y| + \frac{1}{y} + c$$

Q. Solve: $(1+y^2) \cdot dx + x \cdot dy = 0$, given $y(1) = 1$

$\hookrightarrow x = 1, y = 1$

$$(1+y^2)\,dx = -x \cdot dy$$

$$-\frac{dx}{x} = \frac{1}{1+y^2} \cdot dy$$

Integrating both sides: $-\int \frac{dx}{x} = \int \frac{1}{1+y^2}\,dy$

$$-\log|x| = \tan^{-1}(y) + c \quad \text{-----} (*)$$

$$-\log|x| = \tan^{-1}(y) + c \quad \cdots\cdots (*)$$

$x=1,\ y=1 \Rightarrow -\ln|1| = \tan^{-1}(1) + c.$

$\Rightarrow 0 = \pi/4 + c \Rightarrow \boxed{c = -\pi/4}$

Soln: $\boxed{-\log|x| = \tan^{-1}(y) - \pi/4} \longrightarrow$ Particular soln.

(*) For a diff eqn, there can be 2 kinds of solutions:-

(i) General soln: value of c is arbitrary

(ii) Particular soln: value of c can be determined.

(II) Equations Reducible to variable Separation.

Q. Solve: $\dfrac{dy}{dx} = \dfrac{\boxed{x+y}-1}{\boxed{x+y}+1}$

$\int x\cdot dy$ = not defined.

$$(x+y+1)\,dy = (x+y-1)\,dx.$$

$$\boxed{x\cdot dy} + (y+1)\,dy = (x-1)\,dx + \boxed{y\,dx}$$

(Not possible for complete variable separation)

Let $x+y = v.$

Diff: $1 + \dfrac{dy}{dx} = \dfrac{dv}{dx} \Rightarrow \dfrac{dy}{dx} = \left(\dfrac{dv}{dx} - 1\right)$

$$\frac{dv}{dx} - 1 = \frac{v-1}{v+1}$$

$$\frac{dv}{dx} = \frac{v-1}{v+1} + 1 = \frac{v - \cancel{1} + v + \cancel{1}}{v+1} = \frac{2v}{v+1}$$

$$\frac{v+1}{2v}\cdot dv = dx \quad \cdots\cdots \text{[completely variable separated]}$$

$$\frac{v+1}{2v}\cdot dv = dx \quad \ldots\ldots$$ [completely variable separated]

Integrating: $\frac{1}{2}\int \frac{v+1}{v}\,dv = \int dx$

$$\Rightarrow \frac{1}{2}\left[v + \ln|v|\right] = x + C.$$

$$\Rightarrow \boxed{\frac{1}{2}\left[(x+y) + \ln|x+y|\right] = x + C}$$

$$\Rightarrow x+y+\ln|x+y| = 2x+2C.$$

$$y + \ln|x+y| = x + A.$$

$$\ln|x+y| = x - y + A$$

$$\boxed{x+y = ke^{x-y}}$$

(2 different representations of the same soln)