Pg Questions 05.10.23

Thursday, October 5, 2023 7:41 PM

REAL ANALYSIS

Pg Questions...

multiple answers

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \dfrac{a}{1+bx^2}$, $a, b \in \mathbb{R}$, $b \geq 0$. Which of the following are true?

1. f is uniformly continuous on compact intervals of $\mathbb{R}$ for all values of a and b
2. f is uniformly continuous on $\mathbb{R}$ and is bounded for all values of a and b
3. f is uniformly continuous on $\mathbb{R}$ only if $b=0$
4. f is uniformly continuous on $\mathbb{R}$ and unbounded if $a \neq 0$, $b \neq 0$

$b(n) > n$ $\frac{b(n)}{n} > 1$ $a > b$ $\frac{a}{b} > 1$

(2)

Given that $a(n)=\frac{1}{10^{100}}2^n$, $b(n)=10^{100}\log(n)$, $c(n)=\frac{1}{10^{10}n^2}$, which of the following statements are true?

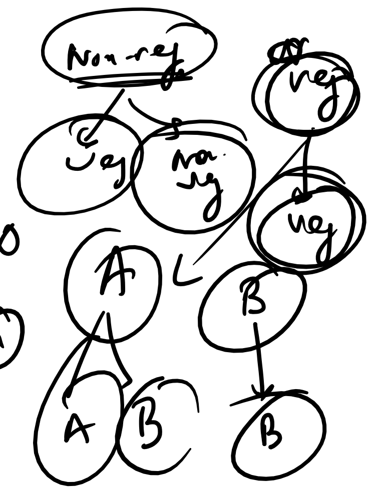
1. $a(n) > c(n)$ for all sufficiently large n
2. $b(n) > c(n)$ for all sufficiently large n
3. $b(n) > n$ for all sufficiently large n
4. $a(n) > b(n)$ for all sufficiently large n

$\lim_{n\to\infty} \frac{a(n)}{c(n)} \to \infty$ $\lim_{n\to\infty} \frac{a(n)}{b(n)} \to \infty$

$a(n) > b(n)$ → Sufficiently large (n)

$a(n) > b(n)$

$\lim_{n\to\infty} \frac{b(n)}{c(n)} = 0$ $\lim_{n\to\infty} \frac{b(n)}{n} = 0$



Sup → LUB

Inf → GLB

$y = \frac{4+x}{x} = \left(\frac{4}{x} + 1\right)$

$y(1) = \frac{4}{1} + 1 = 5$

$y(2) = 3$

$y(3) = 2.33$

$y(4) = 2$

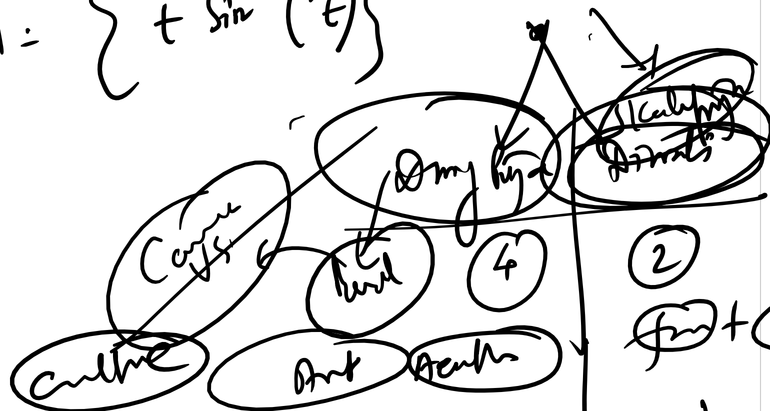
$y(\infty) = 1$

(3)

Let $A = \left\{ t\sin\left(\frac{1}{t}\right) \mid t \in \left(0, \frac{2}{\pi}\right) \right\}$. *Which of the following statements are true?*

1. $\sup(A) < \frac{2}{\pi} + \frac{1}{n\pi}$ for all $n \geq 1$
2. $\inf(A) > \frac{-2}{3\pi} - \frac{1}{n\pi}$ for all $n \geq 1$
3. $\sup(A) = 1$
4. $\inf(A) = -1$

$A : \left\{ t \sin\left(\frac{1}{t}\right) \right\}$



$\sup(A) = \frac{2}{\pi} < \frac{2}{\pi} + \frac{1}{n\pi} \quad \forall n \geq 1$

$\inf(A) = 0 > -\frac{2}{3\pi} - \frac{1}{n\pi} \quad \forall n \geq 1$

$x \geq 3$

$x \geq 4$

$x \leq 10$

$x \geq 3$

$x \geq 4$

(4)

Descarte

$+x^5 - 5x + 2$ — (2) +ve Real Root

Let $f(x) = x^5 - 5x + 2$. Then

1. f has no real root ✗
2. f has exactly one real root ✗
3. f has exactly three real roots ✓
4. all roots of f are real ✗

$f(-x) = -x^5 + 5x + 2$ — (1) −ve Real Root

$5 - 2 - 1 = (2)$ imaginary Roots

Let $f(x, y) = \log(\cos^2(e^{x^2})) + \sin(x + y)$. Then $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x, y)$ is

1. $\frac{\cos(e^{x^2}) - 1}{1 + \sin^2(e^{x^2})} - \cos(x + y)$
2. 0
3. $-\sin(x + y)$
4. $\cos(x + y)$

** (6)

The difference $\log(2) - \sum_{n=1}^{100} \frac{1}{2^n \cdot n}$ is

1. less than 0
2. greater than 1
3. less than $\frac{1}{2^{100} \cdot 101}$ ✓
4. greater than $\frac{1}{2^{100} \cdot 101}$

$$\sum \frac{1}{n \cdot 2^n} = \ln 2$$

$$\ln 2 - \sum \frac{1}{2^n \cdot n}$$

$$= \left(\ln 2 - \sum_{n=1}^{\infty} \frac{1}{2^n \cdot n}\right) + \sum_{n=101}^{\infty} \frac{1}{n \cdot 2^n} = \sum_{n=101}^{\infty} \frac{1}{n \cdot 2^n}$$

$$= \frac{1}{2^{101} \cdot 101} + \frac{1}{2^{102} \cdot 102} + \frac{1}{2^{103} \cdot 103} + \cdots <$$

$$\frac{1}{101 \cdot 2^{101}} + \frac{1}{101 \cdot 2^{102}} + \frac{1}{101 \cdot 2^{103}} + \cdots$$

$$= \frac{1}{101 \cdot 2^{101}} \left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right] = \frac{1}{101 \cdot 2^{101}} \cdot 2$$

$$\frac{a}{1-r} = \frac{1}{1 - 1/2} = \frac{1}{1/2} = 2$$

$$= \frac{1}{2^{100} \cdot 101}$$

(7)

If $\{x_n\}$ is a convergent sequence in $\mathbb{R}$ and $\{y_n\}$ is a bounded sequence in $\mathbb{R}$, then we can conclude that

1. $\{x_n + y_n\}$ is convergent ✗
2. $\{x_n + y_n\}$ is bounded ✓
3. $\{x_n - y_n\}$ has no convergent subsequence ✗
4. $\{x_n + y_n\}$ has no bounded subsequence ✗

Let, $x_n = 1/n$, $y_n = 1/n$

$\Rightarrow \{x_n + y_n\} \rightarrow$ Convergent sequence is Bounded let, $x_n = 1/n$, $y_n = 1/n$

$x_n + y_n = \frac{2}{n} \Rightarrow \{x_n + y_n\} \rightarrow$ Convergent

$\{x_n + y_n\} \rightarrow$ Bounded $\rightarrow$ Every Convt sequence is Bounded

(C + B)

Con + Bou $\rightarrow$ Sequ $\rightarrow$ Sub-sequ (C + B)

$y_n = 2n^2$

$y_n = n^2$

$y\{n\} = 1, 2, 3, 4, 5 \ldots$

$y = \{2n\} = 2, 4, 6, 8 \ldots$

$2n^2 = 2, 8, 18, 32, 50, 72$

$n^2 = 1, 4, 9, 16, 25, 36 \ldots$

$x_n + y_n = \frac{1}{n} + (-1)^n = \frac{1}{n} + 1$ n Even

$= \frac{1}{n} - 1$ n odd

8 Which is Convt, Bounded ..

Bol Weierstrass

Rai Simplification

Given $\{a_n\}$, $\{b_n\}$ two monotone sequences of real numbers and that $\sum a_n b_n$ is convergent, which of the following is true?

1. $\sum a_n$ is convergent and $\sum b_n$ is convergent
2. At least one of $\sum a_n$, $\sum b_n$ is convergent
3. $\{a_n\}$ is bounded and $\{b_n\}$ is bounded
4. At least one of $\{a_n\}$, $\{b_n\}$ is bounded

$a_n = b_n = 1/n$ monotone decreasing

$\sum \frac{1}{n} \cdot \frac{1}{n} = \sum \frac{1}{n^2}$ Convt (p-test)

a_n, b_n is Convt

$a_n = n$ mont incr $\quad b_n = \frac{1}{n^3}$ mont decr

$\sum a_n b_n = 1/n^2 \rightarrow$ Convt

$a_n \rightarrow$ Unbounded

$b_n \rightarrow$ bounded

$b_n \to$ bounded

9/10/11/12/13

8:26 | 9

$5 \times 4 = 20$

✓ +4
X −1
0 +0

Given that there are real constants a,b,c,d such that the identity
$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2$ *holds for all* $x,y \in \mathbb{R}$. *This implies*

1. $\lambda = -5$
2. $\lambda \geq 1$
3. $0 < \lambda < 1$
4. *there is no such* $\lambda \in \mathbb{R}$

Let $f : [-1,1] \to \mathbb{R}$ be a function given by $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Then

1. f is of bounded variation on $[-1,1]$
2. f' is of bounded variation on $[-1,1]$
3. $|f'(x)| \leq 1 \quad \forall\ x \in [-1,1]$
4. $f'(x) \leq 3 \quad \forall\ x \in [-1,1]$

Evaluate $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{n}{k^2+n^2}$

1. $\frac{\pi}{2}$ 2. π 3. $\frac{\pi}{8}$ 4. $\frac{\pi}{4}$

Let $f: \mathbb{R} \to \mathbb{R}$ *be a function satisfying* $f(x+y)=f(x)f(y)$, $\forall\ x,y \in \mathbb{R}$ *and* $\lim_{x \to 0} f(x) = 1$.

Which of the following are necessarily true ?

1. *f is strictly increasing*
2. *f is either constant or bounded*
3. $f(rx)=f(x)^r$ *for every rational* $r \in \mathbb{Q}$
4. $f(x) \geq 0,\ \ \forall\ x \in \mathbb{R}$

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Q}$ the set of all rational numbers. For $0 \le \in \le \frac{1}{2}$, let $A_{\in}$ be the open interval $(0, 1-\in)$. Which of the following are true ?

1. $\sup_{0<\in<\frac{1}{2}} \sup(A_{\in}) < 1$

2. $0 < \in_1 < \in_2 < \frac{1}{2} \Rightarrow \inf(A_{\in_1}) < \inf(A_{\in_2})$

3. $0 < \in_1 < \in_2 < \frac{1}{2} \Rightarrow \sup(A_{\in_1}) > \sup(A_{\in_2})$

4. $\sup (A_{\in} \cap \mathbb{Q}) = \sup (A_{\in} \cap (\mathbb{R} \setminus \mathbb{Q}))$

Let $a_{mn}, m \geq 1, n \geq 1$ be a double array of real numbers. Define

$$P = \liminf_{m \to \infty} \liminf_{n \to \infty} a_{mn},$$

$$Q = \liminf_{m \to \infty} \limsup_{n \to \infty} a_{mn},$$

$$R = \limsup_{m \to \infty} \liminf_{n \to \infty} a_{mn},$$

$$S = \limsup_{m \to \infty} \limsup_{n \to \infty} a_{mn}$$

Which of the following statements are necessarily true ?

1. $P \leq Q$
2. $Q \leq R$
3. $R \leq S$
4. $P \leq S$

Let $S = \{x \in [-1, 4] \mid \sin(x) > 0\}$. *Which of the following is true?*

1. $\inf(S) < 0$
2. *$\sup(S)$ does not exist*
3. $\sup(S) = \pi$
4. *$\inf(S) = \pi/2$*

Let $S = \{f : \mathbb{R} \to \mathbb{R} \mid \exists \in > 0$ such that $\forall \ \delta > 0, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \in\}$. Then

1. $S = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous$\}$
2. $S = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is uniformly continuous$\}$
3. $S = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is bounded$\}$
4. $S = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is constant$\}$

Let $f : (0, \infty) \to \mathbb{R}$ be uniformly continuous. Then

1. $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to \infty} f(x)$ exist.
2. $\lim_{x \to 0^+} f(x)$ exists but $\lim_{x \to \infty} f(x)$ need not exist.
3. $\lim_{x \to 0^+} f(x)$ need not exist but $\lim_{x \to \infty} f(x)$ exists.
4. neither $\lim_{x \to 0^+} f(x)$ nor $\lim_{x \to \infty} f(x)$ need exist.

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers satisfying $a_1 \geq 1$ and $a_{n+1} \geq a_n + 1$ for all $n \geq 1$. Then which of the following is necessarily true?

1. The series $\sum_{n=1}^{\infty} \frac{1}{a_n^2}$ diverges

2. The sequence $\{a_n\}_{n\geq 1}$ is bounded

3. The series $\sum_{n=1}^{\infty} \frac{1}{a_n^2}$ converges

4. The series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges

Let S be the set of $(\alpha, \beta) \in \mathbb{R}^2$ *such that* $\frac{x^{\alpha} y^{\beta}}{\sqrt{x^2 + y^2}} \to 0$ *as* $(x,y) \to (0,0)$. *Then S is contained in*

1. $\{(\alpha, \beta) : \alpha > 0, \beta > 0\}$
2. $\{(\alpha, \beta) : \alpha > 2, \beta > 2\}$
3. $\{(\alpha, \beta) : \alpha + \beta > 1\}$
4. $\{(\alpha, \beta) : \alpha + 4\beta > 1\}$

The equation $11^x + 13^x + 17^x - 19^x = 0$ *has*

1. *no real root*
2. *only one real root*
3. *exactly two real roots*
4. *more than two real roots*

$If\ \lambda_n = \int_0^1 \frac{dt}{(1+t)^n}\ for\ n \in \mathbb{N},\ then$

1. λ_n *does not exist for some n*
2. λ_n *exists for every n and the sequence is unbounded*
3. λ_n *exists for every n and the sequence is bounded*
4. $\lim_{n\to\infty}(\lambda_n)^{1/n} = 1$

For $a,b \in \mathbb{N}$, consider the sequence $d_n = \dfrac{\binom{n}{a}}{\binom{n}{b}}$ for $n>a,b$. Which of the following statements are true ?

1. $\{d_n\}$ converges for all values of a and b
2. $\{d_n\}$ converges if $a < b$
3. $\{d_n\}$ converges if $a = b$
4. $\{d_n\}$ converges if $a > b$

Let $\alpha = 0.1\,0\,1\,1\,0\,1\,1\,1\,0\,1\,1\,1\,1\,0\ldots$ *be a given real number written in base 10, that is, the n-th digit of* α *is 1, unless n is of the form* $\frac{k(k+1)}{2}-1$ *in which case it is 0. Choose all the correct statements from below.*

1. α is a rational number
2. α is an irrational number
3. For every integer $q \geq 1$, there exists an integer $r \geq 1$ such that $\frac{r}{q} < \alpha < \frac{r+1}{q}$.
4. α has no periodic decimal expansion.

Let $A = \{n \in \mathbb{N} : n = 1$ or the only prime factors of n are 2 or 3$\}$, for example, $6 \in A$, $10 \notin A$.

Let $S = \sum_{n \in A} \frac{1}{n}$. Then

1. A is finite
2. S is a divergent series
3. $S = 3$
4. $S = 6$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \dfrac{\sin x}{x} & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$. Then

1. f is not continuous
2. f is continuous but not differentiable
3. f is differentiable
4. f is not bounded