$$\hat{\beta} = \frac{\Sigma(X_i-\bar{X})(Y_i-\bar{Y})}{\Sigma(X_i-\bar{X})^2} = \frac{\Sigma x_i y_i}{\Sigma x_i^2}$$ [Denote $x_i = (X_i - \bar{X})$, $y_i = (Y_i - \bar{Y})$]

$\therefore$ From (ia) $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$ → OLS Estimates.

Digression: Justify why the OLS Estimates $(\hat{\alpha}, \hat{\beta})$ are reasonably good estimates for unknown popln parameters (α, β)

Criteria for a good estimate:

(i) Estimates should be unbiased; i.e we should have $E(\hat{\beta}) = \beta$, $E(\hat{\alpha}) = \alpha$.

(ii) Out of all possible estimates of α, β that can be constructed, OLS Estimates should have low variances.

(iii) As sample size(n) increases, estimates should converge to their popln values -- [Consistency]
$\hat{\beta} \xrightarrow{P} \beta$ and $\hat{\alpha} \xrightarrow{P} \alpha$ as $n \to \infty$.

Properties of OLS Estimators:-

(i) From NE: $\sum_{i=1}^{n}(\underbrace{Y_i - \hat{\alpha} - \hat{\beta}X_i}_{e_i}) = 0 \Rightarrow \Sigma e_i = 0$ --- (ib)

$\sum_{i=1}^{n}(Y_i - \hat{\alpha} - \hat{\beta}X_i)X_i = 0 \Rightarrow \Sigma e_i X_i = 0$ -- (iib)

(ii) From sample: $\text{cov}(X, e) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(e_i - \overset{0}{\bar{e}})$ $\left(\because \bar{e} = \frac{1}{n}\Sigma e_i\right)$

$= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})e_i$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})\, e_i$$

$$= \frac{1}{n}\left[\underbrace{\sum_{i=1}^{n} e_i x_i}_{=0} - \bar{x}\underbrace{\sum_{i=1}^{n} e_i}_{=0}\right] = 0.$$

$\therefore\ cov(X, e) = 0 \Rightarrow corr(X, e) = 0.$

$\therefore$ Explanatory value (x) & error in estimation (e) are uncorrelated.

(iii) To check for unbiaseness of $\hat{\beta}$:-

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{\sum x_i (Y_i - \bar{Y})}{\sum x_i^2} = \frac{\sum x_i Y_i - \bar{Y}\overbrace{\sum x_i}^{=0}}{\sum x_i^2}$$

$x_i = (X_i - \bar{X})$

$\sum x_i = \sum (X_i - \bar{X}) = 0.$

$$\hat{\beta} = \frac{\sum x_i Y_i}{\sum x_i^2} = \sum \Big(\underbrace{\frac{x_i}{\sum x_i^2}}_{=w_i}\Big) Y_i$$

$\Rightarrow \therefore\ \hat{\beta} = \sum w_i Y_i \quad \left[w_i = \frac{x_i}{\sum x_i^2}\right]$ ($\therefore \hat{\beta}$ is a linear combination of Y_i's)

$$\hat{\beta} = \sum w_i (\alpha + \beta X_i + u_i)$$ $[\because$ PRF: $Y_i = \alpha + \beta X_i + u_i]$

$$= \alpha \underbrace{\sum w_i}_{=0} + \beta \underbrace{\sum w_i X_i}_{=1} + \sum w_i u_i$$

$$\boxed{\hat{\beta} = \beta + \sum w_i u_i}$$ ($\because \hat{\beta}$ is a linear combination of u_i's)

Now, $\sum w_i = \sum \frac{x_i}{(\sum x_i^2)} = \frac{1}{(\sum x_i^2)} \cdot \overbrace{(\sum x_i)}^{=0} = 0.$

$$\sum w_i X_i = \sum \frac{x_i \cdot X_i}{(\sum x_i^2)} = \frac{1}{(\sum x_i^2)} \cdot \sum x_i X_i = \frac{\sum (X_i - \bar{X}) \cdot X_i}{\sum (X_i - \bar{X})^2}.$$

$$= \frac{\left(\sum X_i^2 - \bar{X}\sum X_i\right)/n}{\left(\sum (X_i - \bar{X})^2\right)/n}$$

$$= \frac{\frac{1}{n}\sum x_i^2 - \bar{x}^2}{\frac{1}{n}\sum (x_i - \bar{x})^2} = 1 .$$

Now, $\hat{\beta} = \beta + \sum w_i u_i$

$$E(\hat{\beta}) = E\left[\beta + \sum w_i u_i\right].$$

$$= \beta + E\left[\sum w_i u_i\right] \quad (\because \beta \text{ is a fixed constant unknown popln parameter})$$

$$= \beta + \sum w_i \underbrace{E(u_i)}_{=0} \quad (\because w_i\text{'s are non-stochastic}).$$

$\hookrightarrow$ (by assumption)

$\therefore\ E(\hat{\beta}) = \beta$ ..... (unbiased)

(iv) Obtain $Var(\hat{\beta})$:

By defn: $Var(\hat{\beta}) = E\left[\hat{\beta} - E(\hat{\beta})\right]^2$

$$= E\left[\hat{\beta} - \beta\right]^2.$$

$$= E\left[\sum w_i u_i\right]^2 \quad (\because \hat{\beta} = \beta + \sum w_i u_i)$$

$$= E\left[\sum_{i=1}^{n} (w_i u_i)^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} (w_i u_i)(w_j u_j)\right].$$

$$= E\left[\sum w_i^2 u_i^2 + \sum_{i \neq j}\sum w_i w_j u_i u_j\right]$$

$$= \sum w_i^2 \underbrace{E(u_i^2)}_{=\sigma^2} + \sum_{i \neq j}\sum w_i w_j \underbrace{E(u_i u_j)}_{=0}.$$

By assumption:

$Var(u_i) = \sigma^2$.

$\hookrightarrow E(u_i^2) - [\underbrace{E(u_i)}_{=0}]^2 = \sigma^2$

$\therefore\ E(u_i^2) = \sigma^2$

$$= \sum w_i^2 \sigma^2 = \sigma^2 \sum w_i^2$$

$$= \sigma^2 \sum \left(\frac{x_i}{\sum x_i^2}\right)^2$$

$$= \sigma^2 \cdot \frac{1}{(\sum x_i^2)^{\cancel{2}}} \cancel{\left(\sum x_i^2\right)}$$

$\therefore \quad E(u_i^2) = \sigma^2$

$cov(u_i, u_j) = 0$

$E(u_i u_j) - \underbrace{E(u_i)}_{=0}\underbrace{E(u_j)}_{=0} = 0$

$\therefore E(u_i u_j) = 0$

$$= \sigma^2 \cdot \frac{1}{(\sum x_i^2)^{\cancel{2}}} \cancel{(\sum x_i^2)}$$

$$\boxed{Var(\hat{\beta}) = \frac{\sigma^2}{\sum x_i^2}}$$

(v) To check unbiaseness of $\hat{\alpha}$:

We know: $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$

We have shown: $\hat{\beta} = \sum w_i Y_i \quad \left[w_i = \frac{x_i}{\sum x_i^2} \right]$

$$\hat{\alpha} = \frac{1}{n}\sum Y_i - \bar{X}\sum w_i Y_i$$

$$\hat{\alpha} = \sum \Big(\underbrace{\frac{1}{n} - \bar{X} w_i}_{\lambda_i} \Big) Y_i$$

$$\hat{\alpha} = \sum \lambda_i Y_i \quad , \left[\lambda_i = \frac{1}{n} - \bar{X} w_i \right]$$

($\hat{\alpha}$ is a linear combination of Y_i's)

HW.