Sequences

Eg 1: $\{1, 4, 9, 16, \ldots\ldots\} \rightarrow \{x_n\} = \{n^2\}, \quad n = 1, 2, \ldots\ldots$

Eg 2: $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\} \rightarrow \{x_n\} = \{\frac{1}{n}\}, \quad n = 1, 2, \ldots\ldots$

(i) Increasing sequence: $x_{n+1} \geq x_n \ \forall n$.

Strictly increasing sequence: $x_{n+1} > x_n \ \forall n$.

(ii) Decreasing sequence: $x_{n+1} \leq x_n \ \forall n$

Strictly decreasing sequences: $x_{n+1} < x_n \ \forall n$.

(iii) Bounded sequence:

↳ Lower Bound(m): $m \leq x_n \ \forall n, \quad m \in \mathbb{R}$.

↳ Upper Bound (M): $x_n \leq M \ \forall n, \quad M \in \mathbb{R}$.

$\therefore$ A seq $\{x_n\}$ will be bounded if $\exists$, $m, M \in \mathbb{R}$ s.t.

$m \leq x_n \leq M \quad \forall n$

there exists.

Note: $m < x_n \leq M$ or $m < x_n < M$.

m, M are valid bounds of seq $\{x_n\}$.

Eg 1: Increasing seq: $\{x_n\} = \{n^2\} = \{1, 4, 9, \ldots\}$

lower bound   no upper bound.

(*) An increasing is always bounded below. [Lower bound $= x_1$].

Eg 2: Decreasing seq: $\{x_n\} = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ ↳ Lower bound.   Upper bound.

(*) A decreasing seq is always bounded above [Upper bound $= x_1$]

Q. Check if the seq $\{x_n\} = \{\frac{3n+1}{\ldots}\}$ is bounded.

Q. Check if the seq $\{x_n\} = \left\{\frac{3n+1}{n+1}\right\}$ is bounded., $n = 1, 2, 3, \dots$

$$x_{n+1} - x_n = \frac{3(n+1)+1}{(n+1)+1} - \frac{3n+1}{n+1} = \frac{3n+4}{n+2} - \frac{3n+1}{n+1}$$

$$= \frac{(3n+4)(n+1) - (3n+1)(n+2)}{(n+2)(n+1)} > 0$$

$$= \frac{(\cancel{3n^2} + \cancel{3n} + \cancel{4n} + 4) - (\cancel{3n^2} + \cancel{6n} + \cancel{n} + 2)}{(n+2)(n+1)}$$

$$= \frac{2}{(n+2)(n+1)} > 0 \quad \forall n \Rightarrow \text{Strictly increasing seq}$$

$\Rightarrow$ Bounded below.

$\therefore$ Lower bound $x_1 = \frac{3+1}{1+1} = \frac{4}{2} = 2$.

$$x_n = \frac{3n+1}{n+1} = \frac{3(n+1)-2}{(n+1)} = 3 - \left(\frac{2}{n+1}\right)\downarrow < 3 \quad \forall n$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3n+1}{n+1} \quad \left[\frac{\infty}{\infty}\right]$$

L. Hopital : $\lim_{n\to\infty} \frac{3}{1} = 3$. $\Rightarrow x_n < 3 \ \forall n$.

Seq is bounded: $2 \leq x_n < 3 \ \forall n$

Q. The seq $\{x_n\} = \left\{\frac{2n+3}{3n+4}\right\}$ is bounded in:

(a) $\left[\frac{2}{3}, \frac{5}{7}\right]$ ~~(b)~~ $\left(\frac{2}{3}, \frac{5}{7}\right]$ (c) $\left[\frac{2}{3}, \frac{5}{7}\right)$ (d) $\left(\frac{2}{3}, \frac{5}{7}\right)$

$$x_{n+1} - x_n = \frac{2(n+1)+3}{3(n+1)+4} - \frac{2n+3}{3n+4} < 0 \Rightarrow \text{Decreasing seq}$$

$\Rightarrow$ Bounded above.

$\therefore$ Upper bound $x_1 = \frac{2+3}{3+4} = \frac{5}{7}$. $\Rightarrow x_n \leq \frac{5}{7} \ \forall n$.

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$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2n+3}{3n+4} \quad \left[\frac{\infty}{\infty}\right] \quad = \frac{2}{3}.$$

$$\therefore \quad \frac{2}{3} < x_n \leq \frac{5}{7}. \quad (b).$$

Q. $x_n = (-1)^n \dfrac{3n-1}{n}$. Check if x_n is bounded.

$$x_{n+1} - x_n = (-1)^{n+1} \frac{3(n+1)-1}{(n+1)} - (-1)^n \cdot \frac{3n-1}{n}$$

$$= (-1)^n(-1)\frac{3n+2}{n+1} - (-1)^n \frac{3n-1}{n}$$

$$= -(-1)^n \Big[\underbrace{\frac{3n+2}{n+1} + \frac{3n-1}{n}} \Big].$$

$$= (-1)^{n+1} \Big[\;\ldots\ldots\; \Big]$$

1 if $(n+1)$ is even / -1 if $(n+1)$ is odd.

$\therefore$ $x_n = (-1)^n \dfrac{3n-1}{n}$, is neither increasing / nor decreasing.

"Oscillatory sequences"

$$|x_n| = \left|(-1)^n \cdot \frac{3n-1}{n}\right| = |(-1)^n| \cdot \left|\frac{3n-1}{n}\right|$$

$$= \left|\frac{3n-1}{n}\right| = \left|3 - \frac{1}{n}\right| = 3 - \frac{1}{n} < 3.$$

$$\therefore \quad |x_n| < 3 \quad \Rightarrow \boxed{-3 < x_n < 3}$$