03 November 2023 07:34 PM

Expectation / Mean of $x \to E(x)$

1. $x = c = \text{const} \Rightarrow E(x) = c$
2. $y = cx \Rightarrow E(y) = c\,E(x)$
3. $z = x + y$ where x and y both are r.v.s.
   $E(z) = E(x) + E(y)$
4. $y = a + bx \Rightarrow E(y) = a + b\,E(x)$
5. X and Y are indp r.v s, then
   $E(X+Y) = E(X) + E(Y)$ (sum law of expectation)
   and $E(XY) = E(X) \cdot E(Y)$ (product law of expectation)

Variance (v)

$$v(x) = E\left(x - E(x)\right)^2 = E(x^2) - \{E(x)\}^2$$

Properties: if $x = c$ = const $\Rightarrow E(x) = c$

then $v(x) = E\left[x - E(x)\right]^2$
$= E\left[c - c\right]$
$= 0$

$\therefore v(x) = 0$ if $x = c$ = const.

(ii) if $y = bx$ (b const) $\Rightarrow v(y) = b^2 v(x)$

or, $\sigma_y = |b|\,\sigma_x$ (std. deviation)

(iii) if

(iii) if $y = a + bx \rightarrow \boxed{v(y) = 0 + b^2 v(x) = b^2 v(x)}$

deviation

Covariance : $\text{cov}(x,y) = E\{(x - E(x))(y - E(y))\}$

or, $\text{cov}(x,y) = E(xy) - E(x).E(y)$

$\text{cov}(x,y) = \frac{1}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y})$
$= \frac{1}{n}\Sigma xy - \bar{x}\bar{y}$

If x and y are independent r.vs then $\text{cov}(x,y) = 0$

Proof If x and y are indp then using product law of expectation

$E(xy) = E(x) \cdot E(y)$

$\therefore \text{cov}(x,y) = E(xy) - E(x)E(y)$
$= E(x) \cdot E(y) - E(x)E(y)$

$\text{cov}(x,y) = 0$

$[(x+y)^2 = x^2 + y^2 + 2xy]$

Variance of x and y

$V(x+y) = V(x) + V(y) + 2\,\text{cov}(x,y)$
$V(x-y) = V(x) + V(y) - 2\,\text{cov}(x,y)$

if x and y are independent then $\text{cov}(x,y) = 0$

$\therefore$ $V(x+y) = V(x-y) = V(x) + V(y)$

if $z = ax + by$ then $v(z) = v(ax+by)$

$Z = ax + bY$ then $v(Z) = v(ax + bY)$

$$= a^2 v(x) + b^2 v(y) + 2ab \operatorname{cov}(x, y)$$

if $z = ax + by$

$w = ax - by$

$\operatorname{cov}(z, w) = \operatorname{cov}\left[(ax + by), (ax - by)\right]$

$$= \operatorname{cov}\left[a^2x^2 - ab\cancel{x}y + ab\cancel{x}y - b^2y^2\right]$$

$$= a^2 v(x) - b^2 v(y)$$ ✓

Correlation coefficient of x and y $(\rho_{x,y})$

or $r_{x,y}$.

$$r_{x,y} = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y} = \frac{\operatorname{cov}(x, y)}{\sqrt{v(x)}\sqrt{v(y)}}$$ ✓

(*) if x and y are independent

then $\operatorname{cov}(x, y) = 0$

$\Rightarrow r_{x,y} = 0$ ✓

x \ y	0	1	2
0	0.1	0.1	0.1
1	0.1	0.2	0.1
2	0.1	0.1	0.1

<u>Example</u>: Probability distribution of x and y.

Value of x	$P(X = x)$

Value of Y y	$P(Y = y)$

Value of X x	$P(X=x)$
0	0.3
1	0.4
2	0.3
TOTAL	1

Value of Y y	$P(Y=y)$
0	0.4
1	0.3
2	0.3
TOTAL	1

Calculate μ_x, μ_y, σ_x, σ_y, ρ_{xy},

are X and Y independent?

$$E(X) = \sum x_i \cdot p(X = x_i) = 0 \times 0.3 + 1 \times 0.4 + 2 \times 0.3$$
$$= 1$$

$$E(Y) = \sum Y_i P = 0 \times 0.4 + 1 \times 0.3 + 2 \times 0.3$$
$$= 0.3 + 0.6$$
$$= 0.9$$

$$V(X) = E(X^2) - \{E(X)\}^2$$

$$\underline{\underline{E(X^2)}} = \sum x^2 \cdot p = 0^2 \times 0.3 + 1^2 \times 0.4 + 2^2 \times 0.3$$
$$= 0 + 0.4 + 1.2 = \underline{\underline{1.6}}$$

$$V(X) = 1.6 - 1^2$$
$$= 1.6 - 1$$
$$= 0.6$$

$$\therefore \sigma_x = \sqrt{V(X)} = \sqrt{0.6} = 0.7745.$$ ✓

$$V(Y) = E(Y^2) - \{E(Y)\}^2$$

$$E(Y^2) = \sum y^2 p = 0^2 \times 0.4 + 1^2 \times 0.3 + 2^2 \times 0.3$$
$$= 0 + 0.3 + 1.2$$
$$= 1.5$$

$$\therefore V(Y) = 1.5 - 0.9^2$$
$$= 1.5 - 0.81$$
$$= 0.69$$

$$\therefore \delta_Y = \sqrt{0.69} = 0.8306$$

$$\boxed{Cov(X,Y) = E(XY) - E(X)E(Y)}$$

$$E(XY) = \sum XY \cdot P_{ij} = 1 \times 1 \times 0.2 + 1 \times 2 \times 0.1 + 2 \times 1 \times 0.1 + 2 \times 2 \times 0.1$$
$$= 1.$$

$$\therefore Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= 1 - 1 \times 0.9$$

$$Cov(X,Y) = 0.1$$

$$\therefore \rho = \frac{0.1}{0.77 \times 0.83} = \underline{\underline{\text{ans}}}.$$

Q: (i) If $E(X) = 3$, $E(Y) = 5$ } then $E(3X - 5Y + 16) = ?$

(i) If $E(X) = 3$, $E(Y) = 5$ then $E(3X - 5Y + 16) = ?$

(ii) If $E(X) = 4$, $Var(X) = 9$

then $E(X^2) = ?$

(iii) If $E(X) = E(Y) = E(XY) = 1$

then $\rho_{XY} = ?$

(iv) When X and Y are independent with $V(X) = 6$, $V(Y) = 10$

then $V(X - Y) = ?$