Q. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Let $s^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$

Show that s^2 is a consistent estimate of σ^2.

Check: $E(s^2) = \sigma^2$ and $Var(s^2) \to 0$ as $n \to \infty$.

$$s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2.$$

$$E(s^2) = E\left[\frac{1}{n-1}\sum(x_i - \bar{x})^2\right] = \sigma^2.$$

$$Var(s^2) = Var\left[\frac{1}{n-1}\sum(x_i - \bar{x})^2\right].$$

$$= \frac{1}{(n-1)^2} \cdot Var\left[\sum(x_i - \bar{x})^2\right].$$

$$= \frac{1}{(n-1)^2} Var\left[\sigma^2 \cdot \sum\left(\frac{x_i - \bar{x}}{\sigma}\right)^2\right]$$

$$= \frac{\sigma^4}{(n-1)^2} \cdot Var\left[\sum\left(\frac{x_i - \bar{x}}{\sigma}\right)^2\right]$$

$$= \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{(n-1)}.$$

$X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$.

$\left(\frac{X_i - \mu}{\sigma}\right) \sim N(0,1)$.

$\sum\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$

$\hookrightarrow E[\cdot] = n$.

$Var[\cdot] = 2n$.

$\sum\left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi^2_{(n-1)}$

$\hookrightarrow E[\cdot] = (n-1)$

$Var[\cdot] = 2(n-1)$

As $n \to \infty$, $Var(s^2) \to 0$. $\therefore s^2$ is consistent for σ^2.

Q. Let $X_1, X_2, \ldots, X_n$ be a r.s from $U[0, \theta]$. Define $T = \left(\prod x_i\right)^{1/n}$. Show that T is a consistent estimator of (θ/e).

pdf of $U[0,\theta]$ $\quad f(x) = \frac{1}{\theta}, \; 0 \le x \le \theta$.

θ

[Note: $X \sim U[a,b]$ $\quad E(X) = \dfrac{a+b}{2}$, $\quad Var(X) = \dfrac{(a-b)^2}{12}$

$f(x) = \dfrac{1}{b-a}$, $\quad a \le x \le b$.

$$E(X) = \int_a^b x f(x)\,dx$$

$$E(X^2) = \int_a^b x^2 f(x)\,dx$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$T = (x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}$$

$$E(T) = E\left[(x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}\right].$$

$$= E\left[x_1^{\frac{1}{n}} x_2^{\frac{1}{n}} \cdots x_n^{\frac{1}{n}}\right].$$

$$= E(x_1^{1/n}) \cdot E(x_2^{1/n}) \cdots E(x_n^{1/n})$$

$$= \prod E(x_i^{1/n})$$

$$= \prod \frac{\theta^{\frac{1}{n}}}{\left(1+\frac{1}{n}\right)} = \left(\frac{\theta^{\frac{1}{n}}}{1+\frac{1}{n}}\right)^n = \frac{\theta}{\left(1+\frac{1}{n}\right)^n}.$$

$$E(x_i^{\frac{1}{n}}) = \int_0^\theta x_i^{\frac{1}{n}} f(x)\,dx$$

$$= \int_0^\theta x^{\frac{1}{n}} \cdot \frac{1}{\theta}\,dx$$

$$= \frac{1}{\theta} \cdot \int_0^\theta x^{\frac{1}{n}}\,dx$$

$$= \frac{1}{\theta} \cdot \left[\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1}\right]_0^\theta$$

$$= \frac{1}{\cancel{\theta}}\left[\frac{\theta^{\frac{1}{n}+\cancel{1}}}{\frac{1}{n}+1}\right]$$

$$= \frac{\theta^{1/n}}{\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} E(T) = \lim_{n\to\infty} \frac{\theta}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\approx e}} = \left(\frac{\theta}{e}\right)$$

$$T = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}, \quad \lim_{n\to\infty} E(T) \to \left(\frac{\theta}{e}\right)$$

$$E(T^2) = E\left[\prod_{i=1}^{n} x_i\right]^{2/n}.$$

$$= \prod_{i=1}^{n} E\left(x_i^{2/n}\right).$$

$$= \prod \frac{\theta^{2/n}}{\left(1+\frac{2}{n}\right)} = \left(\frac{\theta^{2/n}}{1+\frac{2}{n}}\right)^n$$

$$= \frac{\theta^2}{\left(1+\frac{2}{n}\right)^n}.$$

$$E(x_i^{2/n}) = \int_0^\theta x^{2/n} \cdot \frac{1}{\theta}\,dx$$

$$= \frac{1}{\theta}\left[\frac{x^{\frac{2}{n}+1}}{\frac{2}{n}+1}\right]_0^\theta$$

$$= \frac{1}{\cancel{\theta}}\left[\frac{\theta^{\frac{2}{n}+\cancel{1}}}{\frac{2}{n}+1}\right]$$

$$= \theta^{2/n}$$

$$= \frac{\ \ }{\left(1+\frac{2}{n}\right)^n}$$

$$= \frac{\theta^{2/n}}{\left(1+\frac{2}{n}\right)}$$

$$\lim_{n\to\infty} E(T^2) = \lim_{n\to\infty} \frac{\theta^2}{\underbrace{\left(1+\frac{2}{n}\right)^n}_{\approx e^2}} = \frac{\theta^2}{e^2}.$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

Check: $\lim_{n\to\infty} Var(T) = \lim_{n\to\infty} E(T^2) - \lim_{n\to\infty}[E(T)]^2$

$$= \frac{\theta^2}{e^2} - \frac{\theta^2}{e^2} = 0.$$

Q. Let $X_1, X_2, \cdots, X_n$ be r.s from $f(x) = e^{-(x-\theta)}, x \geq \theta$.

Define $\boxed{T_n = X_{(1)}}$; show $\lim_{n\to\infty} E(T_n) = \theta$.

$X_1 \neq X_{(1)}$.

r.s: $X_1, X_2, \cdots, X_n$ [X_i's can be in any order].

Ordered sample: $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ [Order statistic].

$\therefore X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$.

$X_{(n)} = \max\{X_1, X_2, \cdots, X_n\}$.

$T_n = X_{(1)}$. pdf.

$E(T_n) =$

pdf of $X_{(1)}$:

cdf of $X_{(1)}$: $P[X_{(1)} \leq y]$

$$= P[\min\{X_1, X_2, \cdots, X_n\} \leq y]$$

$$= 1 - P[\min\{X_1, X_2, \cdots, X_n\} > y]$$

$$= 1 - P[X_1 \geq y, X_2 \geq y, \cdots, X_n > y]$$

$$= 1 - P[X_1 \geq y, X_2 \geq y, \ldots, X_n \geq y]$$

$$= 1 - P[X_1 \geq y] \cdot P[X_2 \geq y] \ldots P[X_n \geq y].$$

$$= 1 - \prod P(X_i \geq y).$$

$$= 1 - \{P(X_1 \geq y)\}^n.$$

$$= 1 - \{1 - \underbrace{P(X_1 \leq y)}_{\text{cdf of the popln.}}\}^n.$$

$$f(x) = e^{-(x-\theta)}, \quad x \geq \theta.$$

$$P(X_1 \leq y) = \int_{\theta}^{y} e^{-(x-\theta)}\, dx.$$

HW.

$$\therefore \text{pdf of } X_{(1)} = \frac{d}{dx} F[X_{(1)}] = n[1 - F(x)]^{n-1} f(x).$$