DIFF CALC 2 18.4.2024

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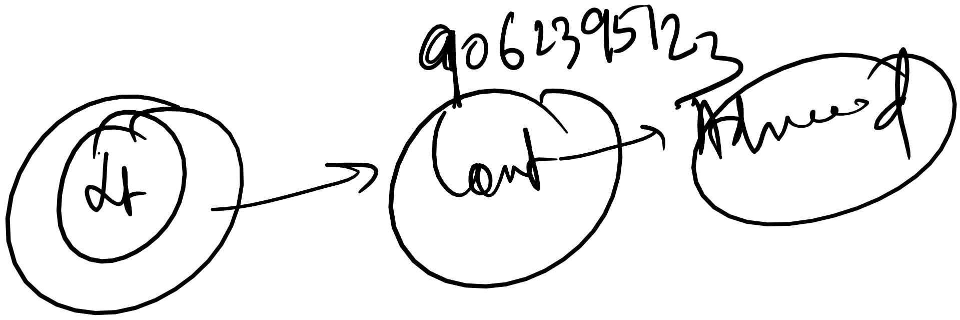

Differential Calculus (1)

A

8. Let $f: R^2 \to R^2$ be defined by $f(x, y) = (e^x \cos(y), e^x \sin(y))$. Then, the number of points in R^2 that do not lie in the range of f is

(a) 0 (b) 1 (c) 2 (d) inf e

16. Let $f(x, y) = e^{x^2+y^2}$ for $(x, y) \in R^2$, and a_n be the determinant of the matrix $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

evaluated at the point $(\cos n, \sin n)$. Then, the limit $\lim_{n \to \infty} a_n$ is

(a) non-existent (b) 0 (c) $6e^2$ (d) $12e^2$

17. Let $f(x, y) = \ln(1 + x^2 + y^2)$ for $(x, y) \in R^2$. Define

$$P = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} \quad Q = \left.\frac{\partial^2 f}{\partial x \, \partial y}\right|_{(0,0)} \Rightarrow \quad R = \left.\frac{\partial^2 f}{\partial y \, \partial x}\right|_{(0,0)} \quad S = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)}$$

Then,

(a) $PS - QR > 0$ and $P < 0$

(b) $PS - QR > 0$ and $P > 0$

(c) $PS - QR < 0$ and $P > 0$

(d) $PS - QR < 0$ and $P < 0$

(a) $PS - QR > 0$ and $P < 0$

(b) $PS - QR > 0$ and $P > 0$

(c) $PS - QR < 0$ and $P > 0$

(d) $PS - QR < 0$ and $P < 0$

25. Let $f(x) = \cos x$ and $g(x) = 1 - \dfrac{x^2}{2}$ for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Then,

(a) $f(x) \geq g(x), \forall x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(b) $f(x) \leq g(x), \forall x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(c) $f(x) - g(x)$ changes sign exactly once on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(d) $f(x) - g(x)$ changes sign more than once on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

28. Let $y : R \to R$ be a twice differentiable function such that y'' is continuous on [0, 1] and $y(0) = y(1) = 0$. Suppose, $y''(x) + x^2 < 0 \; \forall \; x \in [0,1]$. Then,

(a) $y(x) > 0 \; \forall \; x \in (0, 1)$

(b) $y(x) < 0 \; \forall \; x \in (0, 1)$

(c) $y(x) = 0$ has exactly one solution in $(0, 1)$

(d) $y(x) = 0$ has more than one solution in (0, 1)

30. Let $f: R \to R$ be an infinitely differentiable function such that f'' has exactly two distinct zeroes. Then,

(a) f' has atmost 3 distinct zeroes
(b) f' has atleast 1 zero
(c) f has atmost 3 distinct zeroes
(d) f has atleast 2 distinct zeroes

34. Let $f:(-1, 1) \to R$ be a differentiable function satisfying $f(0) = 0$. Suppose, there exists an $M > 0$, such that $|f'(x)| \leq M|x| \; \forall \; x \in (-1, 1)$. Then,

(a) f' is continuous at $x = 0$

(b) f' is differentiable at $x = 0$

(c) $f f'$ is differentiable at $x = 0$

(d) $(f')^2$ is differentiable at $x = 0$

37. Let $f: R^2 \to R$ be defined as follows :

$$f(x,y) = \begin{cases} \dfrac{x^4 y^3}{x^6 + y^6}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Then,

(a) $\lim_{t \to 0} \dfrac{f(t,t) - f(0,0)}{t}$ exists and equals $\dfrac{1}{2}$

(b) $\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)}$ exists and equals 0

(c) $\left.\dfrac{\partial f}{\partial y}\right|_{(0,0)}$ exists and equals 0

(d) $\lim_{t \to 0} \dfrac{f(t,2t) - f(0,0)}{t}$ exists and equals $\dfrac{1}{3}$

44. The value of $\lim_{n\to\infty}\left(n\int_0^1 \frac{x^n}{x+1}\,dx\right)$ is equal to .... (rounded off to two decimal places)

92. The global minimum value of $f(x) = |x-1| + |x-2|^2$ on R is equal to .... (rounded off to two decimal places)

B

12. Let $\theta \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. Consider the functions

$u : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}$ and $v : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}$ given by

$u(x, y) = x - \frac{x}{x^2 + y^2}$ and $v(x, y) = y + \frac{y}{x^2 + y^2}$.

The value of the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ at the point $(\cos\theta, \sin\theta)$ is equal to

(a) $4 \sin\theta$ (b) $4 \cos\theta$ (c) $4 \sin^2\theta$ (d) $4 \cos^2\theta$

The value of the determinant $\begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ at the point $(\cos\theta, \sin\theta)$ is equal to

(a) $4 \sin \theta$ (b) $4 \cos \theta$ (c) $4 \sin^2 \theta$ (d) $4 \cos^2 \theta$

(13)

13. Let $u : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that $u(0) > 0$ and $u'(0) > 0$. Suppose u satisfies $u''(x) = \frac{u(x)}{1 + x^2}$ for all $x \in \mathbb{R}$.

Consider the following two statements:

I. The function uu' is monotonically increasing on $[0, \infty)$.

II. The function u is monotonically increasing on $[0, \infty)$.

Then, which one of the following is correct?

(a) Both I and II are false. (b) Both I and II are true.

(c) I is false, but II is true. (d) I is true, but II is false.

14. The value of $\lim_{n \to \infty} \sum_{k=2}^{n} \frac{\sqrt{n+1}-\sqrt{n}}{k(\ln k)^2}$ is equal to

(a) ∞ (b) 1 (c) e (d) 0

15

15. Let S be the set of all continuous functions $f : [-1, 1] \to \mathbb{R}$ satisfying the following three conditions

(i) f is infinitely differentiable on the open interval $(-1, 1)$,

(ii) the Taylor series $f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots$ of f at 0 converges to $f(x)$ for each $x \in (-1, 1)$,

(iii) $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$

Then, which of the following is/are true?

(a) $f(0) = 0$ for every $f \in S$.

(b) $f'\left(\frac{1}{2}\right) = 0$ for every $f \in S$.

(c) There exists $f \in S$ such that $f'\left(\frac{1}{2}\right) \neq 0$.

(d) There exists $f \in S$ such that $f(x) \neq 0$ for some $x \in [-1, 1]$.

$a_n = \frac{1}{n}$

$f(a_n) = f(0)$

$n \to \infty$, $f\left(\frac{1}{n}\right) \to 0$

$f(0) = 0$

0, 1/4, 1/3, 1/2

0.25, 0.333, 0.5

$f(0) = 0$, $f(1/2) = 0$, $f(1/3) = 0$, $f(1/4) = 0$

$f(a_n) = f(a_{n+1}) = 0$

$b_n \in (a_n, a_{n+1})$

Such that, $f'(b_n) = 0$ — Rolle's theromen

$\therefore b_n = 0$ $f'(b_n) = 0$ $f'(0) = 0$

$c_n \in (b_n, b_{n+1})$

$f''(c_n) = 0$

$c_n \to 0$, $f''(c_n) = f''(0)$

$f(x) = 0$ (Identity = 0)

16. Let S be the set of all functions, $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)|^2 \leq |x - y|^3$ for all $x, y \in \mathbb{R}$.

Then, which of the following is/are true?

(a) Every function in S is differentiable.

(b) There exists a function $f \in S$ such that f is differentiable, but f is not twice differentiable.

(c) There exists a function $f \in S$ such that f is twice differentiable, but f is not thrice differentiable.

(d) Every function is S is infinitely differentiable.

(d) Every function is S is infinitely differentiable.

$f: \mathbb{R} \to \mathbb{R}$ $\quad \lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right|^2 \le |x - y|$

$(f'(x))^2 \le 0$

$|f'(x)|$ Can't be negative

$f'(x) = 0$

$f(x) = \text{Const} \ \forall x$

MSQ

$f(x) = x \quad f(2) = 2 \quad f(3) = 3$

(17)

A real-valued function $y(x)$ defined on $\mathbb{R}$ is said to be periodic, if there exists a real number $T > 0$ such that $y(x + T) = y(x)$ for all $x \in \mathbb{R}$.

Consider the differential equation $\frac{d^2y}{dx^2} + 4y = \sin(ax),\ x \in \mathbb{R}$,

Diff eq

where $a \in \mathbb{R}$ is a constant.

Then, which of the following is/are true?

(a) All solutions of (*) are periodic for every choice of a.

(b) All solutions of (*) are periodic for every choice of $a \in \mathbb{R} - \{-2, 2\}$.

(c) All solutions of (*) are periodic for every choice of $a \in \mathbb{R} - -\{-2, 2\}$.

(d) If $a \in \mathbb{R} - \mathbb{Q}$, then there is a unique periodic solution of (*).

18. The value of the limit

$$\lim_{n\to\infty}\left(\frac{(1^4+2^4+\ldots+n^4)}{n^5}+\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\ldots+\frac{1}{\sqrt{4n}}\right)\right)$$ is equal to .... (rounded off to two decimal places)

$$\int_0^1 x^4\,dx+\int_0^3\frac{1}{\sqrt{1+x}}\,dx$$

$$=\frac{1}{5}+2(2-1)=\frac{1}{5}+2=2.2$$

C

10. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous function satisfying $f(x)=f(x+1)\ \forall\ x\in\mathbb{R}$. Then

(a) there exists infinitely many $x_0\in\mathbb{R}$ such that $f(x_0+\pi)=f(x_0)$

(b) there is no $x_0\in\mathbb{R}$ such that $f(x_0+\pi)=f(x_0)$

(c) f is not necessarily bounded above.

(d) there exists a unique $x_0\in\mathbb{R}$ such that $f(x_0+\pi)=f(x_0)$

Bounded → there exists many $x_0 \in \mathbb{R}$ such that, $f(x_0+\pi) = f(x_0)$

$f(x) = f(x+\pi)$

$f(x) = f(x+1)$

$f(x_0) = f(x_0+\pi)$

$x = x_0$

x, x_0 must be on the neighbourhood

12. Let $D \subseteq \mathbb{R}^2$ be defined by $D = \mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}$. Consider the function $f : D \to \mathbb{R}$ defined by $f(x, y) = x \sin \frac{1}{y}$

Then

(a) f is a continuous function on D and cannot be extended continuously to any point outside D.

(b) f is discontinuous function on D.

(c) f is a continuous function on D and can be extended continuously to the whole of $\mathbb{R}^2$.

(d) f is a continuous function on D and can be extended continuously to $D \cup \{(0, 0)\}$.

13. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in (\mathbb{R} / \mathbb{Q}) \cup \{0\} \\ 1 - \dfrac{1}{p} & \text{if } x - \dfrac{n}{p}, n \in \mathbb{Z} / \{0\}, p \in \mathbb{N} \text{ and } gcd\ (n, p) = 1 \end{cases}$$

then

(a) f is continuous at all $x \in \mathbb{R} / \mathbb{Q}$

(b) f is not continuous at $x = 0$

(c) all $x \in \mathbb{Q} \setminus \{0\}$ are strict local minima for f

(d) f is continuous at all $x \in \mathbb{Q}$

18. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for all $a, b \in \mathbb{R}$ with $a < b$.

$$\frac{f(b) - f(a)}{b - a} = f'\left(\frac{a + b}{2}\right)$$

Then

(a) f is not a polynomial

(b) f must be a linear polynomial

(c) f must be polynomial of degree less than or equal to 2.

(d) f must be a polynomial of degree greater than 2.

32. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function on (a, b). Which of the following statements is/are true?

(a) If $f'(x_0) > 0$ for some $x_0 \in (a, b)$, then there exists a $\delta > 0$ such that $f(x) > f(x_0)$ for all $x \in (x_0, x_0 + \delta)$

(b) If $f'(x_0) > 0$ for some $x_0 \in (a, b)$, then f is increasing in a neighbourhod of x_0.

(c) $f' > 0$ in (a, b) implies that f is increasing in (a, b)

(d) f increasing in (a, b) implies that $f' > 0$ in (a, b)

36. Consider the two function $f(x, y) = x + y$ and $g(x, y) = xy - 16$ defined on $\mathbb{R}^2$. Then

(a) The function g has a global extreme value at (0, 0) subject to the condition $f = 0$

(b) The function g has a global extreme value subject to the condition $f = 0$

(c) The function f has no global extreme value subject to the condition $g = 0$

(d) The function f attains global extreme value at (4, 4) and $(-4, -4)$ subject to the condition $g = 0$

39. Consider the equation $x^{2021} + x^{2020} + \dots + x - 1 = 0$. Then

(a) exactly one real root is positive

(b) no real roots is positive

(c) all real roots are positive

(d) exactly one real root is negative

45. The value of $\lim_{n \to \infty} (3^n + 5^n + 7^n)^{\frac{1}{n}}$ is ....

D

3. Which of the following is FALSE?

(a) $\lim_{x\to\infty} \frac{x}{e^x} = 0$ (b) $\lim_{x\to 0^+} \frac{1}{xe^{1/x}} = 0$ (c) $\lim_{x\to 0^+} \frac{\sin x}{1+2x} = 0$ (d) $\lim_{x\to 0^+} \frac{\cos x}{1+2x} = 0$

6. If the directional derivative of the function $z = y^2 e^{2x}$ at $(2, -1)$ along the unit vector $\boldsymbol{b} = a\hat{i} + \beta\hat{j}$ is zero, then $|\alpha + \beta|$ equals

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) $2\sqrt{2}$

14. Let $a \in \mathbb{R}$. If $f(x) = \begin{cases} (x+a)^2, & x \leq 0 \\ (x+a)^3, & x > 0 \end{cases}$, then

(a) $\frac{d^2f}{dx^2}$ does not exist at $x = 0$ for any value of a

(b) $\frac{d^2f}{dx^2}$ exists at $x = 0$ for exactly one value of a

(c) $\frac{d^2f}{dx^2}$ exists at $x = 0$ for exactly two values of a

(d) $\frac{d^2f}{dx^2}$ exists at $x = 0$ for infinitely many values of a

15. Let $f(x, y) = \begin{cases} x^2 \sin\dfrac{1}{x} + y^2 \sin\dfrac{1}{y}, & xy \neq 0 \\ x^2 \sin\dfrac{1}{x}, & x \neq 0, y = 0 \\ y^2 \sin\dfrac{1}{y}, & y \neq 0, x = 0 \\ 0, & x = y = 0 \end{cases}$

Which of the following is true at (0, 0)?

(a) f is not continuous

(b) $\dfrac{\partial f}{\partial x}$ is continuous but $\dfrac{\partial f}{\partial y}$ is not continuous

(c) f is not differentiable

(d) f is differentiable but both $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are not continuous

17. Let $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$. A point at which the gradient of the function f is equal to zero is

(a) $(-1, 1, -1)$ (b) $(-1, -1, -1)$ (c) $(-1, 1, 1)$ (d) $(1, -1, 1)$

18. The area bounded by the curves $x^2 + y^2 = 2x$ and $x^2 + y^2 = 4x$ and the straight lines $y = x$ and $y = 0$ is

(a) $3\left(\frac{\pi}{2} + \frac{1}{4}\right)$ (b) $3\left(\frac{\pi}{4} + \frac{1}{2}\right)$ (c) $2\left(\frac{\pi}{4} + \frac{1}{3}\right)$ (d) $2\left(\frac{\pi}{3} + \frac{1}{4}\right)$

27. Let $D = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$ and $f : D \to \mathbb{R}$ be a non-constant continuous function. Which of the following is TRUE?

(a) The range of f is unbounded

(b) The range of f is a union of open intervals

(c) The range of f is a closed interval

(d) The range of f is a union of at least two disjoint closed intervals

37. Let $a = \lim_{n \to \infty}\left(\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{(n-1)}{n^2}\right)$ and $b = \lim_{n \to \infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right)$. Which of the following is/are true?

(a) $a > b$ (b) $a < b$ (c) $ab = \ln\sqrt{2}$ (d) $\frac{a}{b} = \ln\sqrt{2}$

41. Let $x_n = n^{\frac{1}{n}}$ and $y_n = e^{1-x_n}$, $n \in \mathbb{N}$. Then the value of $\lim_{n\to\infty} y_n$ is ....

43. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f, f', f'' are continuous functions with $f > 0$, $f' > 0$ and $f'' > 0$. Then $\lim\limits_{x \to -\infty} \dfrac{f(x) + f'(x)}{2}$ is ....

44. Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and $f : S \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Then

$$\max\left\{\delta : \left|x - \frac{1}{3}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{3}\right)\right| < 1\right\}$$

is .... (rounded off to two decimal places)

51. *The sum of the series* $\frac{1}{2(2^2-1)}+\frac{1}{3(3^2-1)}+\frac{1}{4(4^2-1)}+\ldots$ is ……… .

53. The minimum value of the function $f(x, y) = x^2 + xy + y^2 - 3x - 6y + 11$ is ....

54. Let $f(x) = \sqrt{x} + ax, x > 0$ and $g(x) = a_0 + a_1(x - 1) + a_2(x - 1)^2$

be the sum of the first three terms of the Taylor series of $f(x)$ around $x = 1$. If $g(3) = 3$, then α is ....