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(IV) Linear Differential Equations.

Form: $\frac{dy}{dx} + Py = Q$, P, Q : fns of x.

Eg: (i) $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$; $P = \frac{2x}{1+x^2}$, $Q = \frac{1}{(1+x^2)^2}$

(ii) $x \cdot \frac{dy}{dx} + y = 3x^2 - 2$; $P = \frac{1}{x}$, $Q = \frac{3x^2-2}{x}$.

$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x}\right) \cdot y = \frac{3x^2-2}{x}$

Note: Direct variable separation is not possible for Linear Differential Eqns.

Now, $\frac{dy}{dx} + P \cdot y = Q$.

Suppose we have a function fn $f(x)$, multiplied to the differential eqn:

$$f(x) \cdot \frac{dy}{dx} + f(x) \cdot P \cdot y = Q \cdot f(x)$$

LHS: $f(x) \cdot \frac{dy}{dx} + f(x) \cdot P \cdot y = \frac{d}{dx}[y \cdot f(x)]$

this will facilitate the integration process.

Why? $\frac{d}{dx}[y \cdot f(x)] = Q \cdot f(x)$ ..

Integrating: $\int d[y \cdot f(x)] = \int Q \cdot f(x) \cdot dx$

$\Rightarrow y \cdot f(x) = \int Q \cdot f(x) \cdot dx$.

Determining $f(x)$:

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We want: $f(x)\dfrac{dy}{dx} + f(x)\cdot P\cdot y = \dfrac{d}{dx}[y\cdot f(x)]$

$$f(x)\cdot\cancel{\frac{dy}{dx}} + f(x)\cdot P\cdot y = \cancel{f(x)\frac{dy}{dx}} + y\cdot f'(x)$$

$$f(x)\cdot P\cdot \cancel{y} = \cancel{y}\cdot f'(x)$$

$$P = \frac{f'(x)}{f(x)}.$$

Integrating w.r.t x: $\int P\,dx = \ln f(x)$.

$$\boxed{f(x) = e^{\int P\,dx}}$$ Integrating Factor (IF).

Q1. Solve: $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}y = \dfrac{1}{(1+x^2)^2}$ ---- (i)

Note: $e^{\ln f(x)} = f(x)$

$P = \dfrac{2x}{1+x^2}$, $Q = \dfrac{1}{(1+x^2)^2}$.

$$I.F = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln|1+x^2|} = (1+x^2)$$

$\therefore$ Multiply I.F with diff eqn (i):

$$\underbrace{(1+x^2)\cdot\frac{dy}{dx} + (2x)\cdot y} = \frac{1}{(1+x^2)}.$$

$$\frac{d}{dx}[y\cdot(1+x^2)] = \frac{1}{1+x^2}.$$

Integrating: $\int d\,[y\cdot(1+x^2)] = \int \dfrac{1}{1+x^2}\,dx$

$$y\cdot(1+x^2) = \tan^{-1}x + c.$$

Q. Solve: $(1+y+x^2y)\,dx + (x+x^3)\,dy = 0$.

$$\frac{dy}{dx} + \frac{1}{x}\cdot y = -\frac{1}{x+x^3} \quad \text{--- Linear diff eqn}$$

$$P = \frac{1}{x}, \quad Q = -\frac{1}{x(1+x^2)}$$

$$I.F = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x.$$

Multiplying I.F with Diff eqn:

$$x\cdot\frac{dy}{dx} + y = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}[y\cdot x] = -\frac{1}{1+x^2}$$

Int: $$\int d\,[y\cdot x] = -\int \frac{1}{1+x^2}\,dx.$$

$$yx = -\tan^{-1}x + c.$$

Q. Solve: $$\frac{dy}{dx} - 2y\cos x = -2\sin 2x \quad \text{----} \quad (i)$$

$$P = -2\cos x, \quad Q = -2\sin 2x.$$

$$I.F = e^{-2\int \cos x\,dx} = e^{-2\sin x}.$$

(i) x I.F : $$\frac{d}{dx}\left[y\cdot e^{-2\sin x}\right] = -2\sin 2x\, e^{-2\sin x}.$$

Integrate: $$\int d\left[y\cdot e^{-2\sin x}\right] = -2\int \sin 2x\, e^{-2\sin x}\,dx$$

$$y\cdot e^{-2\sin x} = -4\int \sin x\cos x\, e^{-2\sin x}\,dx.$$

Let $-2\sin x = t$.

$-2\cos x\,dx = dt$.

$$y\,e^{-2\sin x} = 2\cdot\int e^t\left(\frac{t}{-2}\right)\cdot dt = -\int t\cdot e^t\,dt.$$

$$y e^{-2\sin x} = 2\int e^{t}\left(\frac{t}{-2}\right)dt = -\int t\cdot e^{t}\,dt$$

$$y e^{-2\sin x} = -\left[t e^{t} - \int e^{t}\,dt\right]$$

$$y e^{-2\sin x} = -t e^{t} + e^{t} + c$$

$$y e^{-2\sin x} = (2\sin x + 1)\, e^{-2\sin x} + c.$$

$$y = 2\sin x + 1 + c e^{2\sin x}$$

Q. Solve: $\dfrac{dy}{dx} - 2y = \cos 3x$.

$\dfrac{dy}{dx} = (\cos 3x + 2y)$

$dy = (\cos 3x + 2y)\, dx$

$P = -2, \quad Q = \cos 3x$

$\therefore \quad I.F = e^{\int P\,dx} = e^{-2x}$.

(i) $\times$ I.F : $\dfrac{d}{dx}\left[y\cdot e^{-2x}\right] = \cos 3x\, e^{-2x}$.

Integrating: $\int d\left[y\cdot e^{-2x}\right] = \int \cos 3x\, e^{-2x}\,dx$

$$y\cdot e^{-2x} = \int \cos 3x\cdot e^{-2x}\,dx.$$

$$I = \int \underbrace{\cos 3x}_{\text{I}}\ \underbrace{e^{-2x}}_{\text{II}}\,dx = \cos 3x\,\frac{e^{-2x}}{-2} - \int \cancel{+}\sin 3x(3)\cdot\frac{e^{-2x}}{\cancel{+}2}\,dx.$$

$$= -\frac{1}{2}\left(\cos 3x\, e^{-2x}\right) - \frac{3}{2}\int \sin 3x\, e^{-2x}\,dx.$$

$$= -\frac{1}{2}\left(\cos 3x\, e^{-2x}\right) - \frac{3}{2}\left[\sin 3x\,\frac{e^{-2x}}{-2} - \int \cos 3x\,(3)\,\frac{e^{-2x}}{-2}\,dx\right]$$

$$= -\frac{1}{2}\left(\cos 3x\, e^{-2x}\right) - \frac{3}{2}\left[-\frac{1}{2}\sin 3x\, e^{-2x} + \frac{3}{2}\boxed{\int \cos 3x\, e^{-2x}\,dx}\right]$$

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HW.