Interpretation of the Optimal Condition: $\frac{MP_L}{MP_K} = \frac{w}{r}$

Finding the cost fn: $C = c(q)$.

Fix a level of output ($q = \bar{q}$) & find the optimal level of L, K that minimizes cost.

$$\min_{\{L,K\}} \; wL + rK \quad \text{s.t} \quad \bar{q} = q(L,K)$$

Lagrangian: $\mathcal{L} = wL + rK + \lambda[\bar{q} - q(L,K)]$ ... λ = Lagrange Multiplier

FOC:

$$\frac{\partial \mathcal{L}}{\partial L} = 0 \Rightarrow w - \lambda\left(\frac{\partial q}{\partial L}\right) = 0 \Rightarrow w - \lambda \cdot MP_L = 0 \Rightarrow \frac{w}{MP_L} = \lambda$$

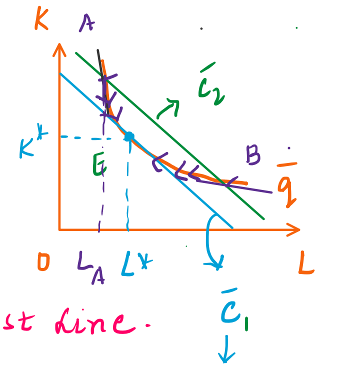
$$\frac{\partial \mathcal{L}}{\partial K} = 0 \Rightarrow r - \lambda\left(\frac{\partial q}{\partial K}\right) = 0 \Rightarrow r - \lambda \cdot MP_K = 0 \Rightarrow \frac{r}{MP_K} = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow \bar{q} - q(L,K) = 0$$

Combining: $\frac{w}{MP_L} = \frac{r}{MP_K} \Rightarrow \frac{MP_L}{MP_K} = \frac{w}{r}$

$\frac{MP_L}{MP_K}$: Abs slope of isoquant (MRTS)

$\frac{w}{r}$: Abs slope of isocost line.



Obj ⇒ Min possible cost ⇒

Graphically ⇒ finding the leftmost isocost line ($\bar{C} = wL + rK$)

Suppose the producer operates at the isocost line ($\bar{C}_2 > \bar{C}_1$) with its aim of producing $\bar{q}$ level of output.

∴ 2 Possible outcomes: pt. A, pt B.

Pt A: Isoquant steeper than isocost.

⇒ |Slope of isoquant| > |Slope of isocost|

$$\Rightarrow \frac{MP_L}{} > \frac{w}{} \Rightarrow MP_L > MP_K$$

$$\Rightarrow \frac{MP_L}{MP_K} > \frac{w}{r} \Rightarrow \boxed{\frac{MP_L}{w} > \frac{MP_K}{r}}$$

MP_L : additional output gained from 1 'L' (benefit)

w : additional cost for employing 1 'L' (cost)

$\frac{MP_L}{w}$: relative benefit of employing 1 extra unit of labour

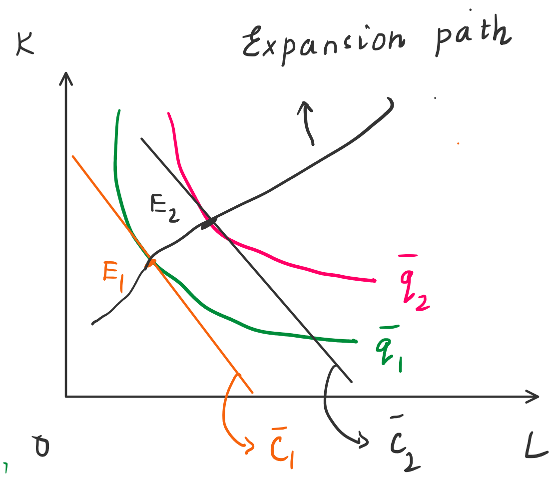
At pt A : $\frac{MP_L}{w} > \frac{MP_K}{r} \Rightarrow$ Incremental relative benefit of labour is higher. [It'll decide to employ more L]

$\therefore$ The firm will try to operate at a pt to the right of pt A.

Similarly for pt 'B', it will be more beneficial to employ more capital. Hence the optimal is stable.

Expansion path

Locus of cost minimizing combinations of (L, K) for every given level of output.



Eg: Suppose prodn fn: $q = L^{1/2} K^{1/2}$

and expansion path of prodn fn $\boxed{K = L}$

Q. Suppose the prodn fn is $q = L + K + L \cdot K$. The producer has a compulsion of producing 6 units of output. Suppose $r = 1$.

Find: (i) degree of homogeneity of the production fn.

(ii) The range of values of w s.t only L is used.

HW (iii) The range of values of w s.t only K is used.

$$q = f(L, K) = L + K + L \cdot K.$$

$$f(\lambda L, \lambda K) = (\lambda L) + (\lambda K) + (\lambda L)(\lambda K).$$

$f(\lambda L, \lambda K) = (\lambda L) + (\lambda K) + (\lambda L)(\lambda K)$.

$= \lambda \cdot L + \lambda K + \lambda^2 L \cdot K$.

$= \lambda [L + K + \lambda \cdot K \cdot L] \neq \lambda^r f(L, K)$.

↳ Not homogeneous of any degree.

(ii) Only L will be used if $\left[\frac{MP_L}{w} > \frac{MP_K}{r}\right]$ ⇒ if it always true.

$(K=0)$

$q = L + K + K \cdot L$, given $\bar{q} = 6$.

$MP_L = 1 + K$. ↳ $K = 0$, $q = 6$ ⇒ $6 = L + 0 + 0$ ⇒ $L = 6$

$MP_K = 1 + L$.

$$\frac{MP_L}{w} > \frac{MP_K}{r} \Rightarrow \frac{1+K}{w} > \frac{1+L}{1}.$$

$$\Rightarrow \frac{1+K}{1+L} > w \Rightarrow w < \frac{1+K}{1+L}$$

Range of w: $w \in [0, \frac{1}{7})$ $\qquad w < \frac{1+0}{1+6} \Rightarrow w < \frac{1}{7}$

Q. A firm can produce output with 2 alternative technologies given by $q = \min\{\frac{K}{3}, \frac{L}{2}\}$ and $q = \min\{\frac{K}{2}, \frac{L}{3}\}$. The marginal cost of production is 20 with both the technologies. Find the expansion path for the prodn fn: $q = K^{2/3} L^{1/3}$.

Prodn fn 1: $q = \min\{\frac{K}{3}, \frac{L}{2}\}$ ⇒ Cost fn: $C_1 =$

Prodn fn 2: $q = \min\{\frac{K}{2}, \frac{L}{3}\}$ ⇒ Cost fn: $C_2 =$

$q = \min\{\frac{K}{3}, \frac{L}{2}\}$

At opt: $\frac{K}{3} = \frac{L}{2} = q$ ⇒ $K^* = 3q$, $L^* = 2q$

$$C_1 = wL^* + rK^* = w(2q) + r(3q) = (2w + 3r)q$$