Revision - Statistics

Q1

Suppose X is a random variable with $P(X > x) = 1/x^2$, for all $x > 1$. The variance of $Y = 1/X^2$ is

(A) 1/4. (B) 1/12. (C) 1. (D) 1/2.

$P(X > x) = \frac{1}{x^2}$

$P(X \leq x) = 1 - P(X > x) = 1 - \frac{1}{x^2} = F(x)$ · [cdf of r.v X]

p.d.f of X : $f(x) = \frac{d}{dx} F(x) = \frac{2}{x^3}$, $x > 1$

$\therefore$ $Var(Y) = Var\left(\frac{1}{X^2}\right) = E\left(\frac{1}{X^4}\right) - \left\{E\left(\frac{1}{X^2}\right)\right\}^2$

$\therefore$ $E\left(\frac{1}{X^4}\right) = \int_1^\infty \frac{1}{x^4} \cdot f(x)\,dx = \int_1^\infty \frac{1}{x^4} \cdot \frac{2}{x^3}\,dx = 2 \int_1^\infty \frac{1}{x^7}\,dx$

$= \frac{2}{-6}\left[x^{-6}\right]_1^\infty = \frac{2}{6} = \frac{1}{3}$.

& $E\left(\frac{1}{X^2}\right) = \int_1^\infty \frac{1}{x^2} f(x)\,dx = \int_1^\infty \frac{1}{x^2} \cdot \frac{2}{x^3}\,dx = 2\int_1^\infty \frac{1}{x^5}\,dx$

$= \frac{2}{-4}\left[x^{-4}\right]_1^\infty = \frac{2}{4} = \frac{1}{2}$.

$\therefore$ $Var(Y) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$ (b).

Q2:

If a fair coin is tossed 5 times, what is the probability of obtaining at least 3 consecutive heads?

(A) 1/8. (B) 5/16. (C) 1/4. (D) 3/16.

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$P[\text{Head}] = P[\text{Tail}] = \frac{1}{2}$.

X : No. of consecutive heads in 5 tosses of a coin

$\therefore P[X \geq 3] = \boxed{P[X=3]} + \boxed{P[X=4]} + \boxed{P[X=5]}$.

$X = 5$: H H H H H $\rightarrow \left(\frac{1}{2}\right)^5$

$X = 4$:
H H H H T
T H H H H
$\} \rightarrow$ 2 cases $\Rightarrow 2 \cdot \left(\frac{1}{2}\right)^5$

$X = 3$
H H H T T/H
T H H H T
T/H T H H H
$\} \rightarrow$ 5 cases $\Rightarrow 5\left(\frac{1}{2}\right)^5$

$\therefore P(X \geq 3) = 5\left(\frac{1}{2}\right)^5 + 2\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 = \frac{8}{2^5} = \frac{1}{4}$ (c)

Q3

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from an exponential distribution with mean λ. If $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are, respectively, the maximum likelihood estimators of the mean and the median of the underlying distribution, then

(A) $\hat{\lambda}_1 < \hat{\lambda}_2$.

(B) $\hat{\lambda}_1 = \hat{\lambda}_2$.

(C) $\hat{\lambda}_1 < \hat{\lambda}_2$ and $\hat{\lambda}_1 > \hat{\lambda}_2$ are both possible.

(D) $\hat{\lambda}_1 > \hat{\lambda}_2$.

$\hat{\lambda}_1 = \hat{\lambda}_{MLE}$.

$\hat{\lambda}_2$ = estimate of median of the exponential distn.

(D) $\hat{\lambda}_1 > \hat{\lambda}_2$.

Compute the median of Exp distn with mean $= \lambda$.

r.v $X \sim Exp\left(\frac{1}{\lambda}\right)$.

pdf of r.v $X \Rightarrow f(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x \geq 0$.

cdf of r.v $X \Rightarrow F(x) = P[X \leq x] = \int_0^x \frac{1}{\lambda} e^{-x/\lambda}\, dx$.

$$= 1 - e^{-x/\lambda}, \quad x \geq 0$$

Denote Median of X be "Med".

$\therefore \quad F(\text{Med}) = 1/2$.

$$1 - e^{-\text{Med}/\lambda} = \frac{1}{2}$$

$$\Rightarrow e^{-\text{Med}/\lambda} = \frac{1}{2} \Rightarrow -\frac{\text{Med}}{\lambda} = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \text{Med} = (\ln 2)\lambda.$$

Estimate of Med $= \hat{\lambda}_2 = (\ln 2)\hat{\lambda}$ ↪ obtained by MLE.

Computing MLE of λ:

$$f(x_i) = \frac{1}{\lambda} e^{-x_i/\lambda}, \quad x_i \geq 0 \ \forall i$$

$$L(\lambda) = \prod_{i=1}^{n} \frac{1}{\lambda} e^{-x_i/\lambda} = \frac{1}{\lambda^n} e^{-\frac{1}{\lambda}\sum x_i}$$

$$l(\lambda) = -n \ln \lambda - \frac{1}{\lambda}\sum x_i$$

For MLE: $\frac{\partial l}{\partial \lambda} = 0 \Rightarrow -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum x_i = 0$

$$\Rightarrow \hat{\lambda}_{MLE} = \bar{X}$$

$\therefore \quad \hat{\lambda}_1 = \bar{X} \quad \Big\} \Rightarrow \hat{\lambda}_1 > \hat{\lambda}_2 \quad (\because \ln 2 < 1)$.

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$\hat{\lambda}_2 = (\ln 2)\bar{X}$ $\Big\} \Rightarrow \hat{\lambda}_1 > \hat{\lambda}_2 \quad (\because \ln 2 < 1)$.

Q4

Suppose X_1 and X_2 are independent and identically distributed exponential random variables with mean 2. The expectation of $\max\{X_1, X_2\}$ is

(A) 4 (B) 3.5 ~~(C)~~ 3 (D) 2.5

$X_1, X_2 \overset{iid}{\sim} Exp(2)$ cdf of X_1:

Define: $Y = \max\{X_1, X_2\} \Rightarrow$ Find $E(Y)$

$\hookrightarrow$ pdf: $g(y) \Rightarrow E(Y) = \int y \cdot g(y)\,dy$.

c.d.f of r.v Y: $G(y) = P[Y \le y]$.

$= P[\max\{X_1, X_2\} \le y]$.

$= P[X_1 \le y, X_2 \le y]$.

$= P[X_1 \le y] \cdot P[X_2 \le y]$.

$= \{P[X_1 \le y]\}^2$.

$\hookrightarrow$ cdf of r.v X_1.

cdf of r.v Y: $G(y) = \{1 - e^{-y/2}\}^2$.

pdf of r.v Y: $g(y) = \dfrac{d\,G(y)}{dy} = e^{-y/2} - e^{-y}, \quad y \ge 0$.

$E(Y) = \int_0^\infty y \cdot g(y)\,dy = \int_0^\infty y\,[e^{-y/2} - e^{-y}]\,dy$

$= 2\int_0^\infty \left(\frac{y}{2}\right) e^{-y/2}\,dy - \int_0^\infty y\,e^{-y}\,dy$.

$$= 2\int_0^\infty y\cdot\left(\frac{1}{2}e^{-y/2}\right)dy - \int_0^\infty y\left(e^{-y}\right)dy$$

exp with mean = 2 exp with mean = 1

$$= 2\cdot 2 - 1 = 3.$$

Q5.

Let X be a random variable with $P(X = 2) = P(X = -2) = 1/6$ and $P(X = 1) = P(X = -1) = 1/3$. Define $Y = 6X^2 + 3$. Then

(A) $\text{Var}(X - Y) < \text{Var}(X)$.

(B) $\text{Var}(X - Y) < \text{Var}(X + Y)$.

(C) $\text{Var}(X + Y) < \text{Var}(X)$.

(D) $\text{Var}(X - Y) = \text{Var}(X + Y)$.

$$f(x) = \begin{cases} -2, & \text{with prob } 1/6. \\ -1, & \text{with prob } 1/3. \\ 1, & \text{with prob } 1/3 \\ 2, & \text{with prob } 1/6. \end{cases}$$

$$g(y) = \begin{cases} 27, & \text{with prob } 1/6 \times 2 \\ 9, & \text{with prob } 1/3 \times 2 \end{cases}$$

$X + Y$:

$X = -2$

$Y = 27$

$Y = 9$

$X = -1$:

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{cov}(X,Y).$$

Now, $\text{cov}(X,Y) = \text{cov}(X, 6X^2+3)$

$$= \text{cov}(X, 6X^2) + \text{cov}(X,3) = 0.$$

$$= \text{cov}(X, 6X^2)$$

$$= E[X\cdot 6X^2] - E(X)\cdot E(6X^2)$$

$$= 6\{\underbrace{E(X^3)}_{=0} - \underbrace{E(X)}_{=0}\cdot E(X^2)\} = 0.$$

$\therefore \quad Var(X+Y) = Var(X) + Var(Y)$

Similarly calculate $\underset{}{\overset{HW}{}}$ $Var(X-Y)$: