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No of solutions of a set of equations

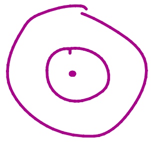
For real numbers $a, b, c, d, a', b', c', d'$, consider the system of equations

$$\begin{aligned} ax^2 + ay^2 + bx + cy + d &= 0 \\ a'x^2 + a'y^2 + b'x + c'y + d' &= 0. \end{aligned}$$

no of solutions of the 2 equations cannot be ____?

If S denotes the set of all real solutions (x, y) of the above system of equations, then the number of elements in S can never be

(a) 0
(b) 1
(c) 2
(d) 3 ✓



$ax^2+ay^2+bx+cy+d=0$

If $a\neq 0$: $x^2+y^2+\left(\frac{b}{a}\right)x+\left(\frac{c}{a}\right)y+\left(\frac{d}{a}\right)=0.$

$x^2+2\cdot\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2+y^2+2\left(\frac{c}{2a}\right)y+\left(\frac{c}{2a}\right)^2-\left(\frac{b}{2a}\right)^2-\left(\frac{c}{2a}\right)^2+\left(\frac{d}{a}\right)=0$

$\left(x+\frac{b}{2a}\right)^2+\left(y+\frac{c}{2a}\right)^2=\left[\left(\frac{b}{2a}\right)^2+\left(\frac{c}{2a}\right)^2-\frac{d}{a}\right]=R^2 \quad -(1)$

centre $=\left(-\frac{b}{2a}, -\frac{c}{2a}\right)$ circle

If $a=0$: $bx+cy+d=0 \quad -(2)$

st line

(1) 2 st lines
- parallel: No soln, 0
- Coincident: infinite soln, ∞
- intersecting: one soln, 1

(2) 2 circles
- Coincident: infinite solns
- Concentric: no solns, 0
- intersecting: 2 solns

(3) 1 st line, 1 circle
- 2 soln
- tangent: 1 soln
- 0 soln

No of solns $0, 1, 2, \infty$.

Application of trigonometric identity to find limit

$-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1$

$0 \leq |\sin x| \leq 1 \qquad 0 \leq |\cos x| \leq 1$

The limit

$$\lim_{x\to 0}\frac{1}{x}\left(\cos(x)+\cos\left(\frac{1}{x}\right)-\cos(x)\cos\left(\frac{1}{x}\right)-1\right)$$

(a) equals 0 .
(b) equals $\frac{1}{2}$.
(c) equals 1.
(d) does not exist.

$$\lim_{x\to 0}\frac{1}{x}\left[\cos x\left\{1-\cos\left(\frac{1}{x}\right)\right\}-\left\{1-\cos\frac{1}{x}\right\}\right]$$

$$=\lim_{x\to 0}\frac{1}{x}\left[\left(1-\cos\frac{1}{x}\right)(\cos x-1)\right].$$

$$=\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)\left(1-\cos\frac{1}{x}\right)$$

$$-1\leq \cos\frac{1}{x}\leq 1$$

$$0\leq 1-\cos\frac{1}{x}\leq 2.$$

$$0\leq \left|1-\cos\frac{1}{x}\right|\leq 2.$$

$$0\leq \left|\left(\frac{\cos x-1}{x}\right)\left(1-\cos\frac{1}{x}\right)\right|\leq 2\left|\frac{\cos x-1}{x}\right|.$$

when $x\to 0$, $\frac{\cos x-1}{x}\to -\sin x\to \sin 0=0$

$$\lim_{x\to 0}\frac{\cos x-1}{x}$$

$$=\lim_{x\to 0}\frac{\frac{d}{dx}(\cos x-1)}{\frac{d}{dx}(x)}$$

$$=\frac{-\sin x}{1}=0$$

$$\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)\left(1-\cos\frac{1}{x}\right)=0$$

Application of factors of a number.

factors of 12 $\rightarrow$ 1, 2, 3, 4, 6, 12.

N. a_1 a_2 a_3 a_4 a_5 a_6.

$$\frac{N}{a_1}=a_6. \qquad \frac{N}{a_2}=a_5 \qquad \frac{N}{a_3}=a_4.$$

$$\boxed{(\text{Product of factors})^2}=(a_1a_2a_3a_4a_5a_6)(a_1a_2a_3a_4a_5a_6)$$

$$=(a_1a_2a_3a_4a_5a_6)\left(\frac{N}{a_6}\frac{N}{a_5}\frac{N}{a_4}\frac{N}{a_3}\frac{N}{a_2}\frac{N}{a_1}\right)=N^6=\boxed{N^{\text{No of factors}}}$$

Let n be a positive integer having 27 divisors including 1 and n, which are denoted by $d_1, \ldots, d_{27}$. Then the product of $d_1, d_2, \ldots, d_{27}$ equals

(a) n^{13}.
(b) n^{14}.
(c) $n^{\frac{27}{2}}$.
(d) $27n$.

27 factors

$$(\text{Product of the factors})^2 = N^{\text{No of factors.}}$$
$$= n^{27}$$
$$\text{Product of the factors} = n^{\frac{27}{2}}$$

$$\boxed{\text{No of maximas and minimas of a function}}$$

Suppose $F : \mathbb{R} \to \mathbb{R}$ is a continuous function which has exactly one local maximum. Then which of the following is true?

(a) F cannot have a local minimum.
(b) F must have exactly one local minimum.
(c) F must have at least two local minima.
(d) F must have either a global maximum or a local minimum.

Finding the value of the power of complex numbers

Suppose $z \in \mathbb{C}$ is such that the imaginary part of z is non-zero and $z^{25} = 1$. Then

$$\sum_{k=0}^{2023} z^k$$

equals

(a) 0 .
(b) 1 .
(c) $-1 - z^{24}$
(d) $-z^{24}$.

$$S = 1 + z + z^2 + z^3 + \cdots + z^{2023}.$$

$$= \frac{z^{2024} - 1}{z - 1} = \frac{\frac{1}{z} - 1}{z - 1}$$

$$81 \times 25 = 2025$$

$$z^{2024} = z^{25 \times 81 - 1} = \frac{z^{25 \times 81}}{z} = \frac{1}{z}$$

$$= \frac{(1 - z)}{z(z-1)}$$

$$= -\frac{1}{z}$$

$$z^{24} . z = 1$$

$$\frac{1}{z} = z^{24}.$$

$$= -z^{24}.$$

Integral of the inverse of a function

$f^{-1}(x) = y.$

$x = f(y)$

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable one-to-one function. If $f(2) = 2$, $f(3) = -8$ and

$$\boxed{\int_2^3 f(x)dx = -3} \rightarrow \int_3^2 f(x)dx = -(-3) = 3$$

$$\int_3^2 f(y)\,dy = 3$$

then

$$\int_{-8}^{2} f^{-1}(x)dx$$

equals

(a) -25 .
(b) 25 .
(c) -31
(d) 31.

$f^{-1}(x) = y.$

$\boxed{x = f(y) \quad dx = f'(y)\,dy.}$

$\boxed{f(2) = 2.}$

$\Rightarrow$ when $x = 2$ $y = 2.$

$\boxed{f(3) = -8}$

when $x = -8$ $y = 3.$

$$I = \int_3^2 y\, f'(y)\,dy.$$

by parts

$y = u.$ $\quad f'(y)\,dy = dv.$

$dy = du$ $\quad \int f'(y)\,dy = v.$

$v = f(y)$

$$I = y\,f(y)\Big|_3^2 - \int_3^2 f(y).dy.$$

$$= 2f(2) - 3f(3) - \int_3^2 f(y)\,dy.$$

$$= 2\times2 - 3\times(-8) - 3 = 4 + 24 - 3 = 25.$$

If $f : [0, \infty) \to \mathbb{R}$ is a continuous function such that

$$f(x) + \ln 2 \int_0^x f(t)dt = 1, x \geq 0,$$

$$\frac{d}{dx}\int_0^x f(t)dt. \quad \frac{d}{dx}F(t)\Big|_0^x \quad \frac{d}{dx}[F(x) - F(0)]$$

const

$= f(x)$

then for all $x \geq 0$,

Convert to a differential equation and then solve for $f(x)$

(a) $f(x) = e^x \ln 2.$
(b) $f(x) = e^{-x} \ln 2.$
(c) $f(x) = 2^x.$
(d) $f(x) = (\frac{1}{2})^x.$

$$f'(x) + \ln 2\, \frac{d}{dx}\int_0^x f(t)dt = 0$$

Integration is the opp of ...

Integration is the opp of differentiation.

$f(x) + x\ln 2 \; \frac{d}{dx}\int_0 f(t)dt = 0$

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$dy = f'(x)\,dx.$

$\int dy = \int f'(x)\,dx.$

$y = f(x).$

$\frac{d}{dx} f(x) = \frac{d}{dx}\int f'(x)\,dx = f'(x)$

$f'(x) + \ln 2 \, f(x) = 0.$

$f(x) = y.$ $\quad y' + \ln 2 \cdot y = 0.$

$\frac{dy}{dx} = -\ln 2 \, y$

$\int \frac{dy}{y} = -\ln 2 \int dx.$

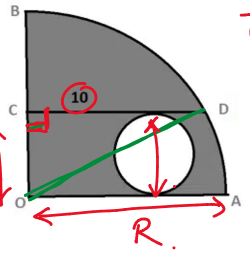
$\ln y = -(\ln 2)x.$

$\ln y = -\ln(2^x) = \ln\left(\frac{1}{2}\right)^x$

$f(x) = y = \left(\frac{1}{2}\right)^x.$

In the following figure, OAB is a quarter-circle. The unshaded region is a circle to which OA and CD are tangents.

$\frac{1}{4}$



If CD is of length 10 and is parallel to OA, then the area of the shaded region in the above figure equals

(a) 25π

(b) 50π

(c) 75π

(d) 100π

$CD^2 + CO^2 = OD^2.$

$OD = OA = R.$

$CO =$ diameter of the circle $= 2r$

Area of the Quadrant $= \frac{1}{4}\pi R^2.$

Area of the circle $= \pi r^2.$

Area of the shaded region

$= \frac{1}{4}\pi R^2 - \pi r^2.$

$\left[10^2 + (2r)^2 = R^2\right] \times \frac{\pi}{4}.$

$25\pi + \pi r^2 = \frac{1}{4}\pi R^2. \rightarrow \frac{1}{4}\pi R^2 - \pi r^2 = 25\pi$

The polynomial $x^{10}+x^5+1$ is divisible by

(a) x^2+x+1.
(b) x^2-x+1.
(c) x^2+1.
(d) x^5-1.

$P(x) = x^{10}+x^5+1$

for $x^5=1$ $\quad P(x) = 1^2+1+1 = 3 \neq 0$.

for $x^2=-1$ $\quad i^{10}+i^5+1$

$x=i$ $\quad = (-1)^5+(-1)^2 i+1 = i \neq 0$.

$\boxed{x^2+x+1} \rightarrow$ cube roots of unity.

$x=\omega$ $\quad \omega^2+\omega+1=0$. $\quad \omega^3=1$

for $x=\omega$ $\quad x^{10}+x^5+1 = \omega^{10}+\omega^5+1$

$=(\omega^3)^3\omega+(\omega^3)\omega^2+1$

$= \omega+\omega^2+1 = 0$

$x^3-1=0 \longrightarrow$ roots are called the cube roots of unity.

$(x-1)(x^2+x+1)=0$.

$\downarrow$

$x = \dfrac{-1\pm\sqrt{1-4}}{2} = \dfrac{-1\pm\sqrt{3}i}{2}$.

$\dfrac{-1+\sqrt{3}i}{2}, \quad \dfrac{-1-\sqrt{3}i}{2}$.