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Random Sampling

$N = 1, 2, \ldots, 100$

$n = 10$ [1, 2, 3, 4, 5, 6 … 10] (10)

[2, 5, 15, … 100] (10)

[3, 9, 17, …] (10)

N^n

$N = 100$

$n = 10$

i=1 → N = 99 + 1

i=2 → 98

i=3

$100 - (2 - 1)$

$N - (i - 1)$

with Replacement (SRSWR)

1. after drawing a member for sample one by one, the member is returned back to population.
2. So at each drawing any member of the population has same probability $1/N$ of being selected.
3. any member of sample can appear more than once in the sample.
4. total no. of samples is N^n, and probab of selecting one sample is $\frac{1}{N^n}$

without Replacement

1. i's not returned back to population after a member is drawn for a sample
2. So at each drawing the probability of selecting a member is $\frac{1}{N-(i-1)}$.
3. No member in the sample is repeated.
4. total no. sample ${}^{N}C_n$, and probability of selecting the sample is $\frac{1}{{}^{N}C_n}$.

* $E(\bar{x})$ —— $SE(\bar{x})$

Expectation and Standard Error of

(Expectation) and Standard error of (Sample mean)

$\hookrightarrow \bar{x}$

Suppose N members of variable $x \rightarrow x_1\ x_2 \ldots x_N$ and we select a random sample of size 'n' from this population and denote the ith member of this sample as x_i (ie $x_1, x_2 \ldots x_n$)

If μ and σ^2 are population mean and population variance

~~then~~ we have,

$$\mu = \frac{1}{N} \sum_{r=1}^{N} X_r \quad \text{--- (1)}$$

$$\sigma^2 = \frac{1}{N} \sum_{r=1}^{N} (x_r - \mu)^2 \quad \text{--- (2)}$$

Again sample mean, $\bar{x} = \sum_{i=1}^{n} x_i / n$ — (3)

Now, $E(\bar{x}) = E\left(\frac{\sum x_i}{n}\right) = \frac{1}{n} \sum_{i=1}^{n} E(x_i)$ — (4)

Further,

$$V(\bar{x}) = V\left(\frac{1}{n}\Sigma x_i\right) = \frac{1}{n^2} V(\Sigma x_i)$$

$$= \frac{1}{n^2}\left[\Sigma V(x_i) + \sum_{i \neq j}\sum Cov(x_i, x_j)\right] \quad (5)$$

Case I **SRSWR**

$$P(x_i = X_r) = \frac{1}{N}$$

$$E(x_i) = \Sigma X_r \, P(x_i = X_r)$$

$$= \sum_{r=1} X_r \times \frac{1}{N} = \frac{1}{N}\sum_r X_r = \mu \quad (6)$$

$$\therefore E(\bar{x}) = \frac{1}{n}\Sigma E(x_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{n}{n} \times \mu = \mu$$

$$\therefore E(x) = E(\bar{x}) = \mu$$ in case of SRSWR.

$$Var(x_i) = E(x_i - \mu)^2 = \sum_{r=1}^{N}(X_r - \mu)^2 P(x_i = X_r)$$

$$= \frac{1}{N}\Sigma(X_r - \mu)^2$$

$$V(x_i) = \sigma^2 \quad (8)$$

$$Cov(x_i, x_j) = E(x_i - \mu)(x_j - \mu)$$

$$= \sum^{N}\sum^{N}(X_r - \mu)(X_s - \mu)P(x_i = X_r, x_j = X_s)$$

$$= \sum_{r=1}^{N} \sum_{s=1}^{N} (X_r - \mu)(X_s - \mu) P(x_i = X_r, x_j = X_s)$$

$$= \frac{1}{N} \underbrace{\sum_{r=1}^{N} (X_r - \mu)} \cdot \frac{1}{N} \underbrace{\sum_{s=1}^{N} (X_s - \mu)}$$

$$= 0 \checkmark \qquad (9)$$

$$\therefore V(\bar{x}) = \frac{1}{n^2} \left[\sum V(x_i) + \sum \sum cov(x_i, x_j) \right]$$

$$= \frac{1}{n^2} \left[\sum_{i=1}^{n} \sigma^2 + 0 \right]$$

$$= \frac{\cancel{n} \sigma^2}{n^{\cancel{2}}}$$

$$V(\bar{x}) = \frac{\sigma^2}{n}$$

$$\therefore SE(\bar{x}) = \sqrt{V(\bar{x})} = \sigma / \sqrt{n} \rightarrow \text{for SRSWR.}$$

Case II : SRSWOR

$$P(x_i = X_r) = \frac{1}{N}$$

$$E(x_i) = \mu \qquad V(x_i) = \sigma^2$$

$$cov(x_i, x_j) = E\left[(x_i - \mu)(x_j - \mu) \right]$$

$$= \sum^{N} \sum^{N} (\quad - \quad)(\quad - \mu) P(x_i = X_r, x_j = X_s)$$

$cov(x_i, x_j)$

$$= \sum_{r=1}^{N}\sum_{s=1}^{N} (x_r-\mu)(x_s-\mu)\, P(x_i = x_r, x_j = x_s)$$

$$= \sum_{r=1}^{N} \sum_{s=1}^{N} (x_r - \mu)(x_s-\mu) \frac{1}{N}\cdot\frac{1}{N-1}$$

$$= \frac{1}{N}\frac{1}{N-1} \sum_{r \neq s}\sum (x_r-\mu)(x_s-\mu)$$

$(x_1+x_2)^2 = x_1^2+x_2^2 + 2x_1x_2$

$2x_1x_2 = (x_1+x_2)^2 - x_1^2 - x_2^2$

$cov \Rightarrow \left(\sum x_i\right)^2 - \left(\sum x_i^2\right)$

$\left\{\sum(x_r-\mu)\right\}^2 - \sum(x_r-\mu)^2$

$$= \frac{1}{N(N-1)}\left[\left\{\sum_{r=1}^{N}(x_r-\mu)\right\}^2 - \sum_{r=1}^{N}(x_r-\mu)^2\right]$$

$$= \frac{1}{N(N-1)}\left[0 - N\sigma^2\right]$$

$$Cov(x_i, x_j) = \frac{1}{\cancel{N}(N-1)}(-\cancel{N}\sigma^2) = \frac{-\sigma^2}{(N-1)}$$

$$\therefore E(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{\cancel{n}\mu}{\cancel{n}} = \mu$$

$$V(\bar{x}) = \frac{1}{n^2}\left[\sum_{i=1}^{n} V(x_i) + \sum_{i \neq j}\sum cov(x_i, x_j)\right]$$

$$= \frac{1}{n^2}\left[n\sigma^2 + \sum_{i\neq j}\sum\left(-\frac{\sigma^2}{N-1}\right)\right]$$

$$= \frac{1}{n^2}\left[n\sigma^2 + n(n-1)\left(\frac{-\sigma^2}{N-1}\right)\right]$$

$$= \frac{\sigma^2}{n} - \frac{(n-1)}{n}\frac{\sigma^2}{(N-1)}$$

$$= \frac{\sigma^2}{n} - \frac{\sigma^2}{n}\frac{(n-1)}{(N-1)}$$

$$= \frac{\sigma^2}{n}\left[1 - \frac{n-1}{N-1}\right]$$

$$= \frac{\sigma^2}{n}\left[\frac{N-1-n+1}{N-1}\right]$$

$$V(\bar{x}) = \frac{\sigma^2}{n}\left[\frac{N-n}{N-1}\right]$$

$$\therefore SE(\bar{x}) = \sqrt{V(\bar{x})} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

Q Marks of 5 students are

20, 24, 20, 30, 26.

If a simple random sample of size 2 is drawn with out replacement. Find the sample mean of each possible samples.

Also find the standard error of the sample mean.

$N = 5$, $n = 2$

$\therefore$ In case of SRSWOR $\Rightarrow$ total no. of samples possible $= {}^{N}C_{n} = {}^{5}C_{2} = 10$

prob distribution of $\bar{x}$

Possible samples are

value of $\bar{x}$	P of $\bar{x}$
20	1/10
22	2/10
23	2/10
25	3/10
27	1/10
28	1/10

Sample value	Sample mean
20 , 24	22
20, 20	20
20, 30	25
20, 26	23
24, 20	22
24, 30	27
24, 26	25
20 , 30	25
20, 26	23
30, 26	28

$$E(\bar{x}) = \mu = \frac{1}{N}\sum x_n = \frac{1}{5}(20+24+20+26+30)$$

$$= \text{ans } 24$$

$$\sigma = \frac{1}{N}\sum(x_n-\mu)^2$$

$$= \frac{1}{5}\{(20-24)^2 + (24-24)^2 + (20-24)^2 + (26-24)^2 + (30-24)^2\}$$

$$V(\bar{x}) = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{n-1}}$$

$= \text{ans } ?$