Solve for $\hat{\alpha}, \hat{\beta}$ using eqns (ia) & (iia):-

Use Cramer's Rule:

$$\hat{\beta} = \frac{\begin{vmatrix} 1 & \bar{Y} \\ \sum X_i & \sum Y_i X_i \end{vmatrix}}{\begin{vmatrix} 1 & \bar{X} \\ \sum X_i & \sum X_i^2 \end{vmatrix}}$$

$$= \frac{\left(\sum Y_i X_i - \bar{Y} \sum X_i\right)/n}{\left(\sum X_i^2 - \bar{X} \sum X_i\right)/n}$$

$$= \frac{\frac{1}{n}\sum X_i Y_i - \bar{X}\bar{Y}}{\frac{1}{n}\sum X_i^2 - \bar{X}^2}$$

$$= \frac{\cancel{\frac{1}{n}} \sum (X_i - \bar{X})(Y_i - \bar{Y})}{\cancel{\frac{1}{n}} \sum (X_i - \bar{X})^2}$$

$$= \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$$

;

$$\frac{\bar{Y} \sum X_i}{n}$$

$$\bar{Y}\Big(\underbrace{\tfrac{1}{n}\sum X_i}_{\bar{X}}\Big)$$

$$= \bar{X}\bar{Y}$$

Notation:

$(X_i - \bar{X}) = x_i$

$(Y_i - \bar{Y}) = y_i$

$$\boxed{\hat{\beta} = \frac{\sum y_i x_i}{\sum x_i^2}}$$

Digression: Sample of 'n' obs: $(Y_i, X_i)_{i=1}^{n}$

Sample covariance of X & Y $= \frac{1}{n}\sum (X_i - \bar{X})(Y_i - \bar{Y})$

Sample variance of X $= \frac{1}{n}\sum (X_i - \bar{X})^2$.

$$\mathrm{cov}(X,Y) = \frac{1}{n}\sum (X_i - \bar{X})(Y_i - \bar{Y}) = \frac{1}{n}\sum \left(X_i Y_i - X_i \bar{Y} - \bar{X} Y_i + \bar{X}\bar{Y} \right)$$

$$= \frac{1}{n}\sum X_i Y_i - \bar{Y}\Big(\underbrace{\tfrac{1}{n}\sum X_i}_{\bar{X}}\Big) - \bar{X}\Big(\underbrace{\tfrac{1}{n}\sum Y_i}_{\bar{Y}}\Big) + \frac{1}{n}\sum \bar{X}\bar{Y}$$

$$= \frac{1}{n}\sum X_i Y_i - \bar{Y}\bar{X} \ldots$$

$$\ldots \underbrace{(\ldots)}_{\bar{x}} \quad \ldots \underbrace{\left(\frac{1}{n}\sum y_i\right)}_{\bar{y}} + \frac{1}{n}\sum \bar{x}\,\bar{y}$$

$$= \frac{1}{n}\sum x_i y_i - \bar{x}\,\bar{y} - \bar{x}\,\bar{y} + \frac{1}{\cancel{n}}\cdot \cancel{n}\cdot \bar{x}\,\bar{y}$$

$$= \frac{1}{n}\sum x_i y_i - \bar{x}\,\bar{y} - \cancel{\bar{x}\,\bar{y}} + \cancel{\bar{x}\,\bar{y}}$$

$$= \frac{1}{n}\sum x_i y_i - \bar{x}\,\bar{y}$$

$$\mathrm{Var}(x) = \frac{1}{n}\sum (x_i - \bar{x})^2 = \frac{1}{n}\sum \left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right)$$

$$= \frac{1}{n}\sum x_i^2 - 2\bar{x}\left(\frac{1}{n}\sum x_i\right) + \frac{1}{n}\sum \bar{x}^2$$

$$= \frac{1}{n}\sum x_i^2 - 2\bar{x}^2 + \frac{1}{\cancel{n}}\cdot \cancel{n}\,\bar{x}^2$$

$$= \frac{1}{n}\sum x_i^2 - 2\bar{x}^2 + \bar{x}^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2 .$$

Result: $\mathrm{cov}(x,y) = \frac{1}{n}\sum (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n}\sum x_i y_i - \bar{x}\,\bar{y}$

$$\mathrm{Var}(x) = \frac{1}{n}\sum (x_i - \bar{x})^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

Now, $\hat{\beta} = \dfrac{\sum x_i y_i}{\sum x_i^2}$

From (ia): $\underset{\text{known}}{\bar{Y}} = \hat{\alpha} + \underset{\text{known}}{\hat{\beta}}\,\underset{\text{known}}{\bar{X}} \Rightarrow \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$.

OLS Estimates: $\hat{\beta} = \dfrac{\sum x_i y_i}{\sum x_i^2}$ and $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$

Note: From (i), (ii):

$$\sum_{i=1}^{n} \overbrace{(Y_i - \hat{\alpha} - \hat{\beta}X_i)}^{e_i} = 0 \;\ldots\ldots (i) \Rightarrow \sum e_i = 0 .$$

$$\sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta}X_i)\,X_i = 0 \;\ldots\ldots (ii) \Rightarrow \sum e_i X_i = 0 .$$

Alternative way of representing the normal equations.

$$\therefore\; \mathrm{cov}(X, e) = \frac{1}{n}\sum (X_i - \bar{X})(e_i - \bar{e}) = \frac{1}{n}\sum (X_i - \bar{X})\,e_i \ldots$$

equations.

$$\therefore \operatorname{cov}(X, e) = \frac{1}{n}\sum(X_i-\bar{X})(e_i-\underbrace{\bar{e}}_{=0}) = \frac{1}{n}\sum(X_i-\bar{X})e_i$$

explanatory variable (X)   error in estimation (e)

$$= \frac{1}{n}\underbrace{\sum X_i e_i}_{=0} - \bar{X}\cdot\frac{1}{n}\underbrace{\sum e_i}_{=0} = 0$$

$\operatorname{cov}(X, e) = 0 \Rightarrow \operatorname{corr}(X, e) = 0$.

$\therefore$ Error is uncorrelated with the explanatory variables.

Properties of the OLS Estimates ($\hat{\alpha}, \hat{\beta}$):-

Note: $\hat{\beta}$ is though to be a good estimate of β.

↳ $\hat{\beta}$ is unbiased estimator of $\beta \Rightarrow E(\hat{\beta}) = \beta$.

↳ $\hat{\beta}$ should have less variance in comparision to other possible estimates of β.

↳ $\hat{\beta}$ should be a consistent estimator of β.

[Consistency of estimate means that as we increase the sample size, the estimator value would gradually tend towards the true population value].

For $\hat{\beta}$:

(I) $\hat{\beta}$ is linear in Y_i:-

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{\sum(Y_i-\bar{Y})x_i}{\sum x_i^2} = \frac{\sum x_i Y_i - \bar{Y}\underbrace{\sum x_i}_{=0}}{\sum x_i^2}$$

$x_i = X_i - \bar{X}$

$\sum x_i = \sum(X_i - \bar{X}) = 0$

$$= \frac{\sum x_i Y_i}{\sum x_i^2} = \sum\Bigg(\underbrace{\frac{x_i}{\sum x_i^2}}_{w_i \text{ (Notation)}}\Bigg)\cdot Y_i = \sum_{i=1}^{n} w_i Y_i$$

$\therefore \hat{\beta} = \sum w_i Y_i$ --- linear combination of Y_i

(II) $\hat{\beta}$ is linear in u_i:

Now: $\hat{\beta} = \sum w_i Y_i$, $w_i = \dfrac{x_i}{\sum x_i^2}$

Now: $\beta = \sum w_i Y_i \quad , \quad w_i = \dfrac{x_i}{\sum x_i^2}$

$$\hat{\beta} = \sum w_i (\alpha + \beta X_i + u_i)$$

$$\hat{\beta} = \alpha \underbrace{\left(\sum w_i\right)}_{=0} + \beta \underbrace{\left(\sum w_i X_i\right)}_{=1} + \sum w_i u_i \Rightarrow \boxed{\hat{\beta} = \beta + \sum w_i u_i}$$

↳ $\hat{\beta}$ is linear in u_i

$$\sum w_i = \sum \frac{x_i}{(\sum x_i^2)} = \frac{1}{(\sum x_i^2)} \underbrace{(\sum x_i)}_{=0} = 0 .$$

$$\sum w_i X_i = \sum \frac{x_i X_i}{\sum x_i^2} = \frac{1}{(\sum x_i^2)} \cdot \sum x_i X_i = \frac{\sum (X_i - \bar{X}) \cdot X_i}{\sum x_i^2}$$

$$= \frac{(\sum X_i^2 - \bar{X} \sum X_i)/n}{(\sum x_i^2)/n}$$

$$= \frac{\frac{1}{n}\sum X_i^2 - \bar{X}^2}{\frac{1}{n}\sum x_i^2}$$

$$= \frac{\cancel{\frac{1}{n}} \sum (X_i - \bar{X})^2}{\cancel{\frac{1}{n}} \sum x_i^2} = \frac{\sum x_i^2}{\sum x_i^2} = 1$$

(III). $\hat{\beta}$ is an unbiased estimator of β:-

$$\hat{\beta} = \beta + \sum w_i u_i$$

$$E(\hat{\beta}) = E\left[\beta + \sum w_i u_i\right]$$

$$= E(\beta) + E\left[\sum w_i u_i\right] .$$

$$= \beta + \sum E(w_i u_i)$$

$$= \beta + \sum w_i \underbrace{E(u_i)}_{=0} = \beta .$$

$w_i = \dfrac{x_i}{\sum x_i^2}$

$\therefore$ w_i are non-stochastic since X_i are non-stochastic.

(IV) Compute the variance of $\hat{\beta}$:

By defn: $Var(\hat{\beta}) = E[\hat{\beta} - E(\hat{\beta})]^2 = E[\hat{\beta} - \beta]^2$

Now, $\hat{\beta} = \beta + \sum w_i u_i \Rightarrow \hat{\beta} - \beta = \sum w_i u_i$

$$\therefore Var(\hat{\beta}) = E[\hat{\beta} - \beta]^2 = E\left[\sum w_i u_i\right]^2$$

$$= E\left[\sum w_i^2 u_i^2 + \sum_{i \neq j}\sum (w_i u_i)(w_j u_j)\right]$$

$$= \sum w_i^2 E(u_i^2) + \sum\sum w_i w_j E(u_i u_j)$$

$$= \sum w_i^2 \underbrace{E(u_i^2)}_{=\sigma^2} + \sum_{i \neq j}\sum w_i w_j \cdot \underbrace{E(u_i u_j)}_{=0} .$$

$$= \sigma^2 \sum w_i^2 = \sigma^2 \sum \left(\frac{x_i}{\sum x_i^2}\right)^2 = \frac{\sigma^2}{(\sum x_i^2)^2} \cancel{(\sum x_i^2)} = \frac{\sigma^2}{\sum x_i^2}$$

Eg: $\left(\sum_{i=1}^{3} a_i\right)^2 = (a_1 + a_2 + a_3)^2$

$$= (a_1^2 + a_2^2 + a_3^2) + 2\,[a_1 a_2 + a_1 a_3 + a_2 a_3] .$$

$$\left(\sum_{i=1}^{3} a_i\right)^2 = \sum_{i=1}^{3} a_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{3} \sum_{j=1}^{3} a_i a_j$$
