Beta and Gamma Functions

Gamma Function

Imp $\quad \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx, \; n > 0.$

Properties:

(i) $\Gamma(n+1) = n\,\Gamma n$

(ii) If n is a positive integer, $\Gamma n = (n-1)!$

(iii) $\Gamma 1 = 1$ and $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

Note: $\Gamma 0$ has no value

$\Gamma -3$ has no value.

Beta Function:

Imp $\quad \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \; ; \quad m, n > 0$

$x = \sin^2\theta$.

Properties:

(i) $\beta(m,n) = \beta(n,m) \quad [m, n > 0]$

(*)(ii) $\beta(m,n) = \dfrac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}$ [Relationship b/w Beta & Gamma Fns]

(iii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta.$

(iv) If m, n are positive integers, then:

$$\beta(m,n) = \frac{\Gamma m \; \Gamma n}{\Gamma(m+n)} = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

$(n-1)!$

Q. Evaluate: $\int_0^\infty e^{-kx} x^{n-1} dx$, $k>0$, n is a positive integer

[original $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx,\ n>0$]

Let $kx = z \Rightarrow dx = \frac{dz}{k}$.

$x=0,\ z=0$

$x\to\infty,\ z\to\infty$

$$\therefore \frac{1}{k}\int_0^\infty e^{-z}\left(\frac{z}{k}\right)^{n-1} dz = \frac{1}{k^n}\int_0^\infty e^{-z} z^{n-1} dz$$

$$= \frac{1}{k^n}\cdot \Gamma n = \frac{(n-1)!}{k^n}$$

Q. Evaluate: $\int_0^\infty e^{-4x} x^{3/2} dx$

$4x = z \Rightarrow dx = \frac{dz}{4}$

$$\therefore \frac{1}{4}\int_0^\infty e^{-z}\left(\frac{z}{4}\right)^{3/2} dz$$

$n+1 = 5/2$

$n = 3/2$

$$\left(\frac{1}{4}\right)^{5/2}\int_0^\infty e^{-z} z^{3/2} dz = \left(\frac{1}{4}\right)^{5/2}\Gamma\left(\frac{5}{2}\right)$$

$\left(\frac{5}{2}-1\right)$

$[\because \Gamma n+1 = n\Gamma n]$

$\Gamma 5/2 = 3/2\ \Gamma 3/2$

$$= \left(\frac{1}{4}\right)^{5/2}\frac{3}{2}\Gamma\frac{3}{2}$$

$$= \left(\frac{1}{4}\right)^{5/2}\left(\frac{3}{2}\right)\overbrace{\left(\frac{1}{2}\right)\Gamma\tfrac{1}{2}}$$

$$= \left(\frac{1}{4}\right)^{5/2}\left(\frac{3}{4}\right)\cdot\sqrt{\pi} = \frac{3\sqrt{\pi}}{128}.$$

Q. Evaluate: $\int_{-\infty}^{+\infty} 5^{-x^2} dx$

$f(x) = 5^{-x^2}$, $f(-x) = 5^{-x^2} = f(x) \Rightarrow$ Even function.

a^x

$\int_{-\infty}^{+\infty} 5^{-x^2} dx = 2\int_0^\infty 5^{-x^2} dx \quad \because \int_{-a}^{+a} \ldots \int_0^a \ldots$

$$\int_{-\infty}^{+\infty} 5^{-x^2}\,dx = 2\int_0^{\infty} 5^{-x^2}\,dx$$

$$= 2\int_0^{\infty} e^{\ln(5^{-x^2})}\,dx$$

$$= 2\int_0^{\infty} e^{-x^2 \ln 5}\,dx.$$

$$\int_{-a}^{+a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

$f(x)$ is Even Fn

$$\int_0^{\infty} e^{-x} x^{n-1}\,dx = \sqrt{n}$$

$$e^{\ln x} = x$$

$$e^{\ln f(x)} = f(x)$$

$$x^2 \ln 5 = z \Rightarrow x = \sqrt{\frac{z}{\ln 5}}$$

$$2x \ln 5\,dx = dz$$

$$dx = \frac{dz}{\ln 5\,(2x)} = \frac{dz}{2\ln 5\sqrt{\frac{z}{\ln 5}}} = \frac{dz}{2\sqrt{\ln 5}\sqrt{z}}.$$

$x = 0,\ z = 0$

$x \to \infty,\ z \to \infty.$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{\ln 5}}\int_0^{\infty} e^{-z}\cdot\frac{1}{\sqrt{z}}\,dz.$$

$$= \frac{1}{\sqrt{\ln 5}}\int_0^{\infty} e^{-z} z^{-1/2}\,dz \qquad \left(\tfrac{1}{2}-1\right)$$

$$= \frac{1}{\sqrt{\ln 5}}\cdot\sqrt{\tfrac{1}{2}} = \sqrt{\frac{\pi}{\ln 5}}$$

Q. Show that: $\displaystyle\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}} \times \int_0^{\pi/2}\sqrt{\cos\theta}\,d\theta = \pi$

$$\frac{1}{4}\cdot\left(2\int_0^{\pi/2}\sin^0\theta\,\cos^{-1/2}\theta\,d\theta\right) \times \left(2\int_0^{\pi/2}\sin^0\theta\cdot\cos^{1/2}\theta\,d\theta\right).$$

$2m-1 = 0 \Rightarrow m = 1/2.$

$2n-1 = -\frac{1}{2} \Rightarrow n = 1/4.$

$2m-1 = 0 \Rightarrow m = 1/2$

$2n-1 = \frac{1}{2} \Rightarrow n = 3/4$

$$\frac{1}{4}\cdot\underbrace{\left[2\int_0^{\pi/2}\sin^{2(\frac{1}{2})-1}\theta\cdot\cos^{2(\frac{1}{4})-1}\theta\,d\theta\right]}\ \underbrace{\left[2\int_0^{\pi/2}\sin^{2(\frac{1}{2})-1}\theta\,\cos^{2(\frac{3}{4})-1}\theta\,d\theta\right]}$$

$$\underbrace{\qquad\qquad}_{\beta\left(\frac{1}{2},\frac{1}{4}\right)} \qquad \underbrace{\int_0 \sin \theta \cos \theta \, d\theta}_{\beta\left(\frac{1}{2},\frac{3}{4}\right)}$$

$$= \frac{1}{4} \cdot \beta\left(\frac{1}{2},\frac{1}{4}\right) \cdot \beta\left(\frac{1}{2},\frac{3}{4}\right)$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$= \frac{1}{4} \cdot \frac{\Gamma \frac{1}{2} \cdot \Gamma \frac{1}{4}}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\right)} \cdot \frac{\Gamma \frac{1}{2} \, \Gamma \frac{3}{4}}{\Gamma\left(\frac{1}{2}+\frac{3}{4}\right)}$$

$$= \frac{1}{4} \cdot \pi \cdot \frac{\Gamma \frac{1}{4} \cdot \cancel{\Gamma \frac{3}{4}}}{\cancel{\Gamma \frac{3}{4}} \cdot \Gamma \frac{5}{4}} = \cancel{\frac{1}{4}} \cdot \pi \cdot \frac{\cancel{\Gamma \frac{1}{4}}}{\cancel{\frac{1}{4}} \cdot \cancel{\Gamma \frac{1}{4}}}$$

$$= \pi \quad (\text{Proved})$$