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Example 1. If z_1 and z_2 are distinct complex numbers such that $|z_1|=|z_2|=1$ and $z_1+z_2=1$, then the triangle in the complex plane with z_1, z_2 and -1 as vertices

(CSIR UGC NET JUNE-2013)

(a) must be equilateral
(b) must be right-angled
(c) must be isosceles, but not necessarily equilateral
(d) must be obtuse angled

$z_1 = a+bi \qquad z_2 = c+di$

$|z_1| = \sqrt{a^2+b^2} = |z_2| = \sqrt{c^2+d^2} = 1$ $\qquad \boxed{a^2+b^2 = c^2+d^2 = 1}$

$z_1+z_2 = 1 \qquad (a+c)+(b+d)i = 1$ $\qquad \boxed{a+c=1} \quad \boxed{b+d=0}$

$b=-d \qquad b^2=d^2$

$a^2+\cancel{b^2} = c^2+\cancel{d^2} \qquad a^2=c^2.$

$(1-c)^2 = c^2 \qquad 1-2c+c^2 = c^2 \qquad \boxed{c=1/2} \quad \boxed{a=\frac{1}{2}}$

$b^2 = 1-a^2 = 1-\frac{1}{4} = \frac{3}{4}$

$d^2 = 1-c^2 = \frac{3}{4}$ $\qquad \boxed{b = \pm\frac{\sqrt{3}}{2}}$

$\boxed{d = \pm\frac{\sqrt{3}}{2}}$

$z_1 = \frac{1}{2}+\frac{\sqrt{3}}{2}i$

$z_2 = \frac{1}{2}-\frac{\sqrt{3}}{2}i$

$z_1 = (1/2, \sqrt{3}/2)$

$z_2 = (1/2, -\sqrt{3}/2)$

$z_3 = (-1, 0)$

$z_1z_2 = \sqrt{0^2+(\sqrt{3})^2} = \sqrt{3}.$

$z_1z_3 = \sqrt{(3/2)^2+(\frac{\sqrt{3}}{2})^2} = \sqrt{3}.$

$z_2z_3 = \sqrt{(3/2)^2+(-\frac{\sqrt{3}}{2})^2} = \sqrt{3}.$

equilateral

$(1+i)^{10}+(1-i)^{10} =$

(a) –1 (b) 1 (c) 0 (d) 2

$1+i = \sqrt{2}e^{i\frac{\pi}{4}}$ $\qquad 1-i = \sqrt{2}e^{i(-\pi/4)}$ $\qquad z = a+ib = re^{i\theta}.$

$(1+i)^{10} = 2^5 e^{i\frac{5\pi}{2}}$ $\qquad (1-i)^{10} = 2^5 e^{-i\frac{5\pi}{2}}.$ $\qquad r = \sqrt{a^2+b^2}, \quad \theta = \tan^{-1}(\frac{b}{a})$

$= 32(\cos\frac{5\pi}{2} + i\sin 5\pi/2)$ $\qquad = 32(\cos(-5\pi/2) + i\sin(-5\pi/2))$

$= 32(0+i)$ $\qquad = 32[0+i(-1)]$

$= 32(0-i)$

If $iz^3 + z^2 - z + i = 0$, then find $|z|$.

$-1 = i^2$

$iz^3 + z^2 + i^2 z + i = 0$

$z^2(iz+1) + i(iz+1) = 0$

$(z^2+i)(iz+1) = 0.$

$z^2 = -i \qquad z = -\frac{1}{i}$

$z^2 = 0 + i(-1)$

$z^2 = e^{i3\pi/2} \qquad z = -\frac{i}{i^2} = \frac{-i}{-1} = i = e^{i\pi/2}.$

$z = e^{i3\frac{\pi}{4}}. \qquad r = 1 = |z|$

$r = 1 = |z|$

If $z = \sqrt{2i}$, then z is equal to

(a) $\pm\frac{1}{\sqrt{2}}(1+i)$ *(b)* $\pm\frac{1}{\sqrt{2}}(1-i)$ *(c)* $\pm(1-i)$ ✓*(d)* $\pm(1+i)$

$2i = 1 + 2i - 1 = 1 + 2i + i^2 = (1+i)^2.$

$\sqrt{2i} = \pm(1+i)$

$z = \sqrt{ai} \longrightarrow$ when $a = \pm 2$.

$ai = \left(\frac{a}{2}\right)^2 + 2\times\left(\frac{a}{2}\right)\times i + i^2 - \left(\frac{a^2}{4} + i^2\right)$

$= \left(\frac{a}{2} + i\right)^2 - \left(\frac{a^2}{4} - 1\right)$

Exercise 6. The principal value of $\log(i^{\frac{1}{4}})$ *is*

(a) $i\pi$ *(b)* $\frac{i\pi}{2}$ *(c)* $\frac{i\pi}{4}$ ✓*(d)* $\frac{i\pi}{8}$

$i\theta$

Exercise 6. The principal ...

(a) $i\pi$ (b) $\frac{i\pi}{2}$ (c) $\frac{i\pi}{4}$ (d) $\frac{i\pi}{8}$

$\log(i^{1/4}) = \frac{1}{4}\log i$

$= \frac{1}{4} \times i\frac{\pi}{2} = \frac{i\pi}{8}$

$z = re^{i\theta}$

$\log z = \log r + i\theta$.

$i = 1 \times e^{i\pi/2}$.

$\log i = \log 1 + i\frac{\pi}{2}$.

$= i\frac{\pi}{2}$.

Example 5. Let P(z), Q(z) be two complex non-constant polynomials of degree m, n respectively. The number of roots of P(z)=P(z)Q(z) counted with multiplicity is equal to: *(CSIR UGC NET JUNE – 2016)*

(a) min {m,n} (b) max {m,n}

(c) m+n (d) m-n

$P(z) = a_0 z^{n} + a_1 z^{n-1} + \cdots + a_n$.

$Q(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_n$.

$P(z) - P(z)Q(z) = 0$.

$P(z)[1 - Q(z)] = 0$. $\Rightarrow P(z) = 0$ or $Q(z) = 1$.

$\downarrow$ n roots. $\downarrow$ m roots.

Example 8. Consider the polynomial $f(z) = z^2 + az + p^{11}$, where $a \in \mathbb{Z}\setminus\{0\}$ and $p \geq 13$ is a prime. Suppose that $a^2 \leq 4p^{11}$. Which of the following statements is true?

(a) f has a zero on the imaginary axis

(b) f has a zero for which the real and imaginary parts are equal

(c) f has distinct roots

(d) f has exactly one real root

(CSIR UGC NET DEC-2019)

a = integer $\neq 0$

(c) f has distinct roots
(d) f has exactly one real root

$a =$ integer $+$

(CSIR UGC NET DEC-2019)

$z = \frac{-a \pm \sqrt{a^2 - 4p^{11}}}{2}$

$D = a^2 - 4p^{11}$

$z = -\frac{a}{2} \pm \frac{\sqrt{D}}{2} i$

$a^2 - 4p^{11} \leq 0$

$a^2 - 4p^{11} < 0$ ⇒ ~~real~~ No roots

2 distinct complex roots

$a^2 = 4p^{11}$ ⇒ 2 equal roots

$a = 2p^{11/2}$

$= 2p^5\sqrt{p}$

irrational

Common roots of the equation $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is/are

(a) ω, ω^2 (b) $1, \omega$ (c) $1, \omega^2$ (d) $1, \omega, \omega^2$

$\omega^3 = 1$

$z = 1$ ✗

$\omega^3 + 2\omega^2 + 2\omega + 1$

$= \omega^3 + \omega^2 + \omega + \omega^2 + \omega + 1$

$= \omega(\omega^2 + \omega + 1) + (\omega^2 + \omega + 1)$

$= 0 + 0. = 0.$

$\omega^{1985} + \omega^{100} + 1$

$= \omega^2 + \omega + 1$

$= 0.$

$\omega^2 + \omega + 1 = 0$

$(\omega^2)^2 + \omega^2 + 1$

$= \omega^4 + \omega^2 + 1$

$= \omega + \omega^2 + 1 = 0$

Complex form of $\sqrt{3+4i}$ is

(a) $\sqrt{3+i}$ (b) $2-i$ (c) $2+i$ (d) $\sqrt{3}-i$

$2 \quad \ldots \quad ^2 = (2+i)^2$

Complex form of $\sqrt{3+4i}$ is

(a) $\sqrt{3+i}$ (b) $2-i$ (c) $2+i$ (d) $\sqrt{3}-i$

$3+4i = 2^2 + 2\times2\times i + i^2 = (2+i)^2$.

If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = 1$, then $|z_1+z_2+z_3|$ is

(a) equal to 1 (b) less than 1 (c) greater than 3 (d) equal to 3

$|z|^2 = z\bar{z}$

$z = a+bi$. $\bar{z} = a-bi$

$z\bar{z} = a^2 - b^2i^2 = a^2+b^2 = \left(\sqrt{a^2+b^2}\right)^2$

$= |z|^2$.

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

$\frac{1}{z_1} = \frac{\bar{z}_1}{|z_1|^2} = \bar{z}_1$

$\frac{1}{z_2} = \bar{z}_2$ $\frac{1}{z_3} = \bar{z}_3$

$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = |\bar{z}_1+\bar{z}_2+\bar{z}_3|$

$= |\overline{z_1+z_2+z_3}| = 1$

Logarithms

2 The value of $\log_4[\log_2\{\log_2(\log_3 81)\}]$ is equal to

(a) –1 (b) 0 (c) 1 (d) 2

4 If $a = \log_3 5$, $b = \log_{17} 25$, which one of the following is correct?

(a) $a < b$ (b) $a = b$ (c) $a > b$ (d) None of these

$a = \frac{\log 5}{\log 3}$ $b = \frac{\log 25}{\log 17} = \frac{2\log 5}{\log 17}$.

$2\log 3 < \log 17$.

$9 < 17$.

(a) $a < b$ (b) $a = b$ (c) $\cancel{a > b}$ (d) None of these

$a = \frac{\log 5}{\log 3}$

$b = \frac{\log 25}{\log 17} = \frac{2\log 5}{\log 17}$

$2\log 3 < \log 17$

$9 < 17$

$\frac{1}{2\log 3} > \frac{1}{\log 17}$

1 If $\log_{0.16}(a+1) < \log_{0.4}(a+1)$, then a satisfies

(a) $\cancel{a > 0}$ (b) $0 < a < 1$ (c) $-1 < a < 0$ (d) None of these

$\log_{0.16}(a+1) < \log_{0.4}(a+1)$

$\frac{\log(a+1)}{2\cancel{\log 0.4}} < \frac{\log(a+1)}{\cancel{\log 0.4}}$

$\log(a+1) < \log(a+1)^2$

$a+1 < a^2+2a+1$

$a^2 + a > 0$

$a(a+1) > 0$

4 The value of $\log_{10} 3$ lies in the interval

(a) $\left(\frac{2}{5}, \frac{1}{2}\right)$ (b) $\left(0, \frac{1}{2}\right)$ (c) $\left(0, \frac{2}{5}\right)$ (d) None of these

$\log_{10} 9 \approx 1$

$2\log_{10} 3 \approx 1$

$\log_{10} 3 \approx \frac{1}{2}$

$\log_{10} x = \frac{2}{5}$

$x = (100)^{1/5}$

$2^5 = 32$

$3^5 = 243$

$32 < 100 < 243$

$2^5 < 100 < 3^5$

13. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$, $z = 1 + \log_c ab$, then $\left(\frac{xyz}{xy + yz + zx}\right)$ is equal to

$\log_a(abc)$ $\log_b(abc)$ $\log_c(abc)$

(a) 0 (b) 1

(c) -1 (d) 2

$\frac{1}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$

$\frac{1}{x} = \log_{abc} a$

$\frac{1}{y} = \log_{abc} b$

$\frac{1}{z} = \log_{abc} c$

$\log_{abc} abc = 1$