LINEAR ALGEBRA 28.3.2024

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(A)

definite matrix $n \times n \to$ symmetric $X^T A X > 0$ $X \to n \times 1$ matrix

$x \neq 0$

$\frac{dy}{dx} > 0$ ✓ strictly increasing

$\frac{dy}{dx} \geq 0$ increasing ..

+ve definite

$X^T A X \geq 0$ +ve semi-definite

< 0 -ve definite

≤ 0 -ve semi-definite

10^{10}

$2C_2$

Linear Algebra U...

X^0

$T: W \to W$ $T(p(x)) = p'(x)$

$B = \{1, x, x^2, x^3\}$

1. Let W be the vector space of all real polynomials of degree at most 3. Define $T: W \to W$ by $(Tp)(x) = p'(x)$, where p' is the derivative of p. The matrix of T in the basis $\{1, x, x^2, x^3\}$, considered as column vectors, is given by

   1. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
   2. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$
   3. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
   4. $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$

$T(x) = 1 = 1 \cdot x^0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$

$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$

$T(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3x^2 + 0 \cdot x^3$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Let $S = \{A : A = [a_{ij}]_{5\times5}, a_{ij} = 0 \text{ or } 1 \ \forall i, j, \sum_j a_{ij} = 1 \ \forall i \text{ and } \sum_i a_{ij} = 1 \ \forall j\}$. Then the number of elements in S is

   1. 5^2 2. 5^5 3. $5!$ 4. 55

$$A \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \text{we can interchange Rows ..}$$

$5! \to 120$

3. Let ξ be a primitive fifth root of unity. Define $A = \begin{pmatrix} \xi^{-2} & 0 & 0 & 0 & 0 \\ 0 & \xi^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & \xi^2 \end{pmatrix}$

   For a vector $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5$, define $|v|_A = \sqrt{|vAv^T|}$, where v^T is transpose of v. If $w = (1, -1, 1, 1, -1)$, then $|w|_A$ equals

   1. 0 2. 1 3. -1 4. 2

4. The dimension of the vector space of all symmetric matrices of order $n \times n$ $(n \geq 2)$ with real entries and trace equal to zero is

   1. $\left(\frac{n^2-n}{2}\right)-1$ 2. $\left(\frac{n^2+n}{2}\right)-1$ 3. $\left(\frac{n^2-2n}{2}\right)-1$ 4. $\left(\frac{n^2+2n}{2}\right)-1$

$n = 3$ 2

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

dim $\to$ 6

$3 \times 3 \to 6$

dim $= 6 - 1 = 5$

5. Let D be a non-zero $n \times n$ real matrix with $n \geq 2$. Which of the following implications is valid?

   X 1. $\det(D) = 0$ implies rank $(D) = 0$ 2. $\det(D) = 1$ implies rank $(D) \neq 1$

   X 3. $\operatorname{rank}(D) = 1$ implies $\det(D) \neq 0$ 4. $\operatorname{rank}(D) = n$ implies $\det(D) \neq 1$ X

i) $D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ det $\to 0$ $R \to 1$

ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ det 1 $R \neq 1$

iii) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $R = 1$ $d = 0$

iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R = 2 = n$ det $= 1$

$2P_2$

6. Suppose A, B are $n\times n$ positive definite matrices and I be the $n\times n$ identity matrix. Then which of the following are positive definite.

1. $A+B$ 2. ABA 3. A^2+I 4. AB

7. Let N be a 3×3 non-zero matrix with the property $N^3=O$. Which of the following is/are true? → nilpotent matrix

1. N is not similar to a diagonal matrix. ✓ A non zero nilpotent matrix ⇒ non diagonalizable
2. N is similar to a diagonal matrix. ✗
3. N has one non-zero eigenvector. ✓
4. N has three linearly independent eigenvectors. ✗

①③

$N \to 0$ an eigenvalue, an all 0 so, N has 3 independent eigen vectors.

GM → 3

AM → 3

at least 1 non-zero EV corresponding to $\lambda=0$, but N can't have 3 linearly independent vectors

[If A has 3 LI EV then GM = AM of 0 = 3]

8. Let $a_{ij}=a_ia_j,\ 1\le i,j\le n$, where $a_1,\dots,a_n$ are real numbers. Let $A=((a_{ij}))$ be the $n\times n$ matrix (a_{ij}). Then

1. it is possible to choose $a_1,\dots,a_n$ so as to make the matrix A non-singular.
2. the matrix A is positive definite if $(a_1,\dots,a_n)$ is a non-zero vector.
3. the matrix A is positive semidefinite for all $(a_1,\dots,a_n)$.
4. for all $(a_1,\dots,a_n)$, zero is an eigenvalue of A.

9. Let T be a linear transformation on the real vector space $\mathbb{R}^n$ over $\mathbb{R}$ such that $T^2=\lambda T$ for some $\lambda\in\mathbb{R}$. Then

1. $\|Tx\|=|\lambda|\,\|x\|$ for $x\in\mathbb{R}^n$.
2. If $\|Tx\|=\|x\|$ for some non-zero vector $x\in\mathbb{R}^n$, then $\lambda=\pm1$
3. $T=\lambda I$, where I is the identity transformation on $\mathbb{R}^n$.
4. If $\|Tx\|>\|x\|$ for a non-zero vector $x\in\mathbb{R}^n$, then T is necessarily singular.

(1,7) JAM 2024

10. Let M be the vector space of all $\boxed{3\times3}$ real matrices and let $A=\begin{pmatrix}2&1&0\\0&2&0\\0&0&3\end{pmatrix}$. Which of the following are subspaces of M?

1. $\{X\in M: XA=AX\}$ ✓
2. $\{X\in M: X+A=A+X\}$ ✓
3. $\{X\in M: \mathrm{trace}(AX)=0\}$
4. $\{X\in M: \det(AX)=0\}$ ✗

x^2+2, x^2+2 3

$(\alpha X+\beta Y)A = \alpha XA+\beta YA$
$= \alpha AX+\beta AY$
$= A(\alpha X+\beta Y)\in S_1$

11. Let $W=\{p(B): p$ is a polynomial with real coefficients$\}$, where $B=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$. The dimension d of the vector space W satisfies

1. $4\le d\le6$
2. $6\le d\le9$
3. $3\le d\le8$
4. $3\le d\le4$

$S_3=\{x\in M: \mathrm{tr}(Ax)=0$

$x,y\in S_3,\ \alpha,\beta\in\mathbb{R}$

$\mathrm{tr}(A(\alpha x+\beta y)) = \alpha\,\mathrm{tr}(Ax)+\beta\,\mathrm{tr}(Ay)=0+0=0$

$\alpha x+\beta y\in S_3\ \forall \alpha,\beta$

12. The determinant of the matrix $\begin{bmatrix}1&0&0&0&0&2\\0&1&0&0&2&0\\0&0&1&2&0&0\\0&0&2&1&0&0\\0&2&0&0&1&0\\2&0&0&0&0&1\end{bmatrix}$ is

1. 0
2. -9
3. -27
4. 1

option D

$X=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ $Y=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$AX=$ $AY=$

$AX+AY$
$=\begin{bmatrix}2&1&0\\0&2&0\\0&0&3\end{bmatrix}\neq0$

13. For a positive integer n, let P_n denote the space of all polynomials $p(x)$ with coefficients in $\mathbb{R}$ such that $\deg p(x)\le n$, and let B_n denote the standard basis of P_n given by $B_n=\{1,x,x^2,\dots,x^n\}$. If $T:P_3\to P_4$ is the linear transformation defined by $T(p(x))=x^2p'(x)+\int_0^x p(t)dt$ and $A=(a_{ij})$ is the 5×4 matrix of T with respect to standard bases B_3 and B_4, then

1. $a_{32}=\frac{3}{2}$ and $a_{33}=\frac{7}{3}$
2. $a_{32}=\frac{3}{2}$ and $a_{33}=0$
3. $a_{32}=0$ and $a_{33}=\frac{7}{3}$
4. $a_{32}=0$ and $a_{33}=0$

14. Let A be a 5×4 matrix with real entries such that the space of all solutions of the linear system $AX^t=[1,2,3,4,5]^t$ is given by $\{[1+2s,2+3s,3+4s,4+5s]^t: s\in\mathbb{R}\}$. (Here, M^t denotes the transpose of a