SOURAV SIR'S CLASSES

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Application of infinite GP series.

$S = 1 + x + x^2 + x^3 + \cdots$ to infinite terms

$= \dfrac{1}{1-x}.$

Let S be the set of those real numbers x for which the identity

$$\sum_{n=2}^{\infty} \cos^n x = (1+\cos x)\cot^2 x$$

is valid, and the quantities on both sides are finite. Then

(a) S is the empty set

(b) $S = \{x \in \mathbb{R} : x \neq n\pi \text{ for all } n \in \mathbb{Z}\}$

(c) $S = \{x \in \mathbb{R} : x \neq 2n\pi \text{ for all } n \in \mathbb{Z}\}$

(d) $S = \{x \in \mathbb{R} : x \neq (2n+1)\pi \text{ for all } n \in \mathbb{Z}\}$

LHS $= \sum_{n=2}^{\infty} \cos^n x$

$= \cos^2 x + \cos^3 x + \cos^4 x + \cdots$

$\cos = 0$ $(4n+1)(\pi/2)$

II $\cos$ -ve

I $\cos$ +ve

$\cos = -1$ $(2n+1)\pi$.

$2n\pi$ $\cos = 1$

III $\cos$ -ve

IV $\cos$ +ve

$(4n+3)\pi/2$. $\cos = 0$

$x = 2n\pi$. ✗

$\cos x = 1$

LHS $= 1 + 1 + \cdots$

diverging series

$x = (2n+1)\pi$. ✗

$\cos x = -1$

LHS $= 1 - 1 + 1 - 1 \cdots$

diverging series

$x \neq$ even multiple of π

$x \neq$ odd " " π.

LHS $= \dfrac{\cos^2 x}{1-\cos x} = \dfrac{(1+\cos x)\cos^2 x}{(1+\cos x)(1-\cos x)} = \dfrac{(1+\cos x)\cos^2 x}{1-\cos^2 x}$

$= (1+\cos x)\dfrac{\cos^2 x}{\sin^2 x} = (1+\cos x)\cot^2 x. = \text{RHS}$

$S = 1+1+1+\cdots$ diverging.

Derivatives of composite functions using chain Rule

$h(x) = f(g(x)).$

$h'(x) = f'(g(x))\, g'(x)$ chain Rule.

Let f, g be continuous functions from $[0, \infty)$ to itself

$x \geq 0$

$$h(x) = \int_{2^x}^{3^x} f(t)dt, x > 0 \quad (1)$$

and

$$F(x) = \int_0^{h(x)} g(t)dt, x > 0. \quad (2)$$

If F' is the derivative of F, then for $x > 0$.

(a) $F'(x) = g(h(x))$.
(b) $F'(x) = g(h(x))\left[f\left(3^x\right) - f\left(2^x\right)\right]$.
(c) $F'(x) = g(h(x))\left[x3^{x-1}f\left(3^x\right) - x2^{x-1}f\left(2^x\right)\right]$.
(d) $F'(x) = g(h(x))\left[3^x f\left(3^x\right)\ln 3 - 2^x f\left(2^x\right)\ln 2\right]$. ✓

$\frac{d}{dx}\int_0^{h(x)} g(t)\,dt = \frac{d}{dx} g(h(x))$

differentiating (2).

$F'(x) = g'(h(x)).h'(x)$

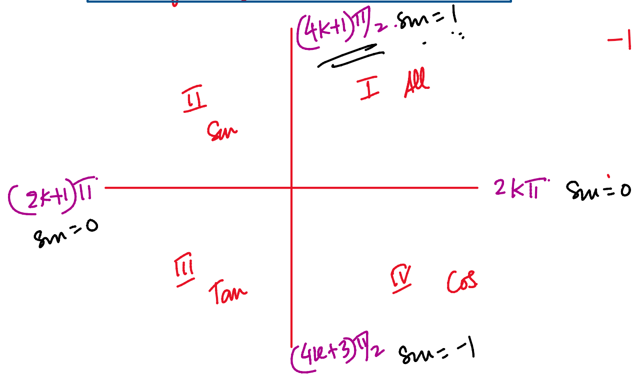
differentiating (1)

$h'(x) = f'(3^x).3^x \ln 3 - f'(2^x).2^x \ln 2.$

$\frac{d}{dx}\int_{2^x}^{3^x} f(t)\,dt = \frac{d}{dx}\left[f(3^x) - f(2^x)\right] = f'(3^x).3^x\ln 3 - f'(2^x).2^x\ln 2.$

$F'(x) = g'(h(x))\left[3^x f'(3^x)\ln 3 - 2^x f'(2^x)\ln 2\right]$

$\boxed{\text{Solving trigonometric equations}}$



$-1 \leq \sin x/\cos x \leq 1$

$(4k+1)\frac{\pi}{2} = 2k\pi + \frac{\pi}{2}.$

Let

$$S = \left\{\left(\theta \sin\frac{\pi\theta}{1+\theta}, \frac{1}{\theta}\cos\frac{\pi\theta}{1+\theta}\right) : \theta \in \mathbb{R}, \theta > 0\right\}$$

$x \quad , \quad y.$

and

$$T = \left\{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, xy = \frac{1}{2}\right\}.$$

How many elements does $S \cap T$ have?

(a) 0

(b) 1 ✓

(c) 2

(d) 3

$x = \frac{1}{3}\sin\frac{\pi\,1/3}{4/3} = \frac{1}{3}\sin\frac{\pi}{4}$

$= \frac{1}{3}\cdot\frac{1}{\sqrt{2}} = \frac{1}{3\sqrt{2}}$

$S. \leftarrow x, y \rightarrow T.$

$x = \theta \sin\frac{\pi\theta}{1+\theta} \qquad y = \frac{1}{\theta}\cos\frac{\pi\theta}{1+\theta}. \qquad y = \frac{1}{3}\cos\frac{\pi}{4} = \frac{1}{3\sqrt{2}}.$

$\frac{\theta}{1+\theta} = \frac{+ve}{+ve} = +ve > 0 \qquad xy = \frac{1}{2}.$

$0 < 1+\theta, \quad \frac{\theta}{1+\theta} < 1$

$\cancel{\theta}\sin\frac{\pi\theta}{1+\theta}\cdot\frac{1}{\cancel{\theta}}\cos\frac{\pi\theta}{1+\theta} = \frac{1}{2}$

$2\sin\theta\cos\theta = \sin 2\theta.$

$2\sin\frac{\pi\theta}{1+\theta}\cos\frac{\pi\theta}{1+\theta} = 1$

$k = -\frac{1}{4} + \frac{\theta}{1+\theta}. \quad \& \quad k = 0$

$\sin\frac{2\pi\theta}{1+\theta} = 1 = \sin\left[\frac{\pi}{2} + 2k\pi\right]$

$\frac{\theta}{\ } = \frac{1}{4}$

$k = -\frac{1}{4} + \frac{\theta}{\ }$

$k = -\frac{1}{4} + \frac{\theta}{1+\theta}$. at $\theta = 0$

$\sin \frac{2\pi\theta}{1+\theta} = 1 = \sin\left[\frac{\pi}{2} + 2k\pi\right]$

$\frac{\theta}{1+\theta} = \frac{1}{4}$

$4\theta = 1+\theta$

$3\theta = 1$

$\boxed{\theta = \frac{1}{3}}$

$0 < \frac{\theta}{1+\theta} < 1$

$-\frac{1}{4} < k < \frac{3}{4}$

$\underline{k=0}$

$k = -\frac{1}{4} + \frac{\theta}{1+\theta}$

$k = 1 - \frac{1}{4}$

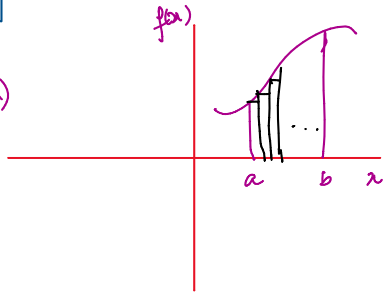
$= \frac{3}{4}$

$\frac{2\pi\theta}{1+\theta} = \frac{\pi}{2} + 2k\pi \rightarrow k \in \mathbb{Z}$

$\frac{\theta}{1+\theta} = \frac{\pi}{4\pi} + \frac{2k\pi}{2\pi} = \frac{1}{4} + k$

Finding the limit of a sum using integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} h\,f(a+kh)$$



The limit

$$\lim_{n\to\infty} n^{-\frac{3}{2}} \left((n+1)^{(n+1)}(n+2)^{(n+2)}\dots(2n)^{(2n)}\right)^{\frac{1}{n^2}}$$

equals

(a) 0

(b) 1

(c) $e^{-\frac{1}{4}}$

(d) $4e^{-\frac{3}{4}}$ ✓

$x = (n^{-3/2})\left[(n+1)^{n+1}(n+2)^{n+2}\dots\dots(n+n)^{n+n}\right]^{\frac{1}{n^2}}$ x/n in the power use log.

$\ln x = -\frac{3}{2}\ln n + \frac{1}{n^2}\ln\left[(n+1)^{n+1}(n+2)^{n+2}\dots\dots \text{to n terms}\right]$

$= -\frac{3}{2}\ln n + \frac{1}{n^2}\left[(n+1)\ln(n+1) + (n+2)\ln(n+2)\dots\dots \text{to n terms}\right]$

$\frac{n+1}{n} = 1 + \frac{1}{n}$.

$\ln x = -\frac{3}{2}\ln n + \frac{1}{n}\left[\left(1+\frac{1}{n}\right)\left\{\ln n + \ln\left(1+\frac{1}{n}\right)\right\} + \left(1+\frac{2}{n}\right)\left\{\ln n + \ln\left(1+\frac{2}{n}\right)\right\} + \dots\right]$ to n terms

$\ln(n+1)$

$= \ln\left[n\left(1+\frac{1}{n}\right)\right]$

$= \ln n + \ln\left(1+\frac{1}{n}\right)$

$\ln x = -\frac{3}{2}\ln n + \frac{1}{n}\sum_{k=1}^{n}\left[\left(1+\frac{k}{n}\right)\left\{\ln n + \ln\left(1+\frac{k}{n}\right)\right\}\right]$

$2\ln 2 = \ln 2^2 = \ln 4$

$\ln x = -\frac{3}{2}\ln n + \frac{1}{n}\ln n \sum_{k=1}^{n}\left(1+\frac{k}{n}\right) + \frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)\ln\left(1+\frac{k}{n}\right)$

$\frac{1}{4}x^2\Big|_1^2 = \frac{1}{4}(2^2-1^2)$

$= \frac{3}{4}$.

$\ln x = -\frac{3}{2}\ln n + \frac{1}{n}\ln n\left[n + \frac{1}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)}{2}\right] + \frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)\ln\left(1+\frac{k}{n}\right)$

$\lim_{n\to\infty}\ln x = 2\ln 2 - \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^2$ ✓ $= \ln n\left[-\frac{3}{2} + \frac{1}{n}\left\{n + \frac{n}{2} + \frac{1}{2}\right\}\right] + \frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)\ln\left(1+\frac{k}{n}\right)$

$= \ln 4 - \frac{1}{4}(4-1)$

$= \ln n\left[-\frac{3}{2} + 1 + \frac{1}{2} + \frac{1}{2n}\right] + \frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)\ln\left(1+\frac{k}{n}\right)$

$\int \ln x \cdot f(x)\,dx$.

$= \ln 4 - \frac{3}{4}$

$\ln\left(\lim_{n\to\infty} x\right) = \ln\left(4e^{-3/4}\right)$

$\ln x = \lim_{n\to\infty}\frac{\ln n}{2n} + \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)\ln\left(1+\frac{k}{n}\right)$

$1 + \frac{k}{n} = x$.

$\boxed{\ln x = u}$

$\boxed{f(x)\,dx = dv}$

$x = 4e^{-3/4}$. $n\to\infty$.

$k=0 \quad x=1$

$k=n \quad x=2$.

$\lim_{n\to\infty}\ln x = 0 + \int_1^2 x\ln x\,dx$.

$\frac{1}{x}dx = du$

$\lim_{n\to\infty}\ln x = \frac{x^2}{2}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{dx}{x}$

$x\ln x = u$

$x\,dx = dv$

$du = \frac{1}{x}dx$

$v = \frac{x^2}{2}$.

$v = \int f'(x)\,dx = f(x)$

Application of $e^{i\theta} = \cos\theta + i\sin\theta$

$\lim_{n\to\infty} \ln x$ $v = \frac{x^2}{2}$ $x\,dx = dv$ $du = \frac{1}{x}dx$ $V = \int f'(x)dx = f(x)$

$\boxed{\text{Application of } e^{i\theta} = \cos\theta + i\sin\theta}$

Complex   Real   imaginary part

The value of

$$\sum_{k=0}^{202}(-1)^k \binom{202}{k} \cos\left(\frac{k\pi}{3}\right)$$

equals

(a) $\sin\left(\frac{202}{3}\pi\right)$.
(b) $-\sin\left(\frac{202}{3}\pi\right)$
(c) $\cos\left(\frac{202}{3}\pi\right)$.
(d) $\cos^{202}\left(\frac{\pi}{3}\right)$.

$\frac{k\pi}{3} = \theta$.

$\sum (-1)^k \, {}^{202}C_k \cos\theta$.

$\Downarrow$

Real part

real part in the expansion of

$\sum (-1)^k \, {}^{202}C_k e^{i\theta}$

$$\sum_{k=0}^{202} (-1)^k \, {}^{202}C_k e^{i\theta}.$$

$$\sum_{k=0}^{n} (-1)^k \, {}^{n}C_k e^{kx} = (1-e^x)^n.$$

$x = \frac{i\pi}{3}$

$$= \cos\left(\frac{202\pi}{3}\right)$$

$$= \cos\left(\frac{-202\pi}{3}\right)$$

$$\sum_{k=0}^{202} (-1)^k \, {}^{202}C_k e^{\frac{ik\pi}{3}}$$

$$= \left(1 - e^{\frac{i\pi}{3}}\right)^{202} = \left(e^{-i\pi/3}\right)^{202} = e^{-\frac{202\pi}{3}i}$$

$$= \cos\left(\frac{-202\pi}{3}\right) + i\sin\left(\frac{-202\pi}{3}\right)$$

Real part = ?

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$e^{\frac{i\pi}{3}} + e^{-\frac{i\pi}{3}} = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$$

$$e^{-i\pi/3} = 1 - e^{i\pi/3}$$