Numbers

$\mathbb{N} = \{1, 2, 3, \cdots\}$, Null

$W = \{0, 1, 2, 3, \cdots\}$

$\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$ $\curvearrowleft \mathbb{Z}^+$

$2 + \_\_ = 2$ ✗

$3 - 4 = ?$ ✗

$4 \times \_\_ = 1$ ✗

$x, -x$

Additive inverse : number + (additive inverse) = additive identity

number + additive identity = number
(zero)

Multiplicative identity : number × MI = number

$\mathbb{Q} = \{ p/q,\ p, q \in \mathbb{Z},\ q \neq 0 \}$ $\longrightarrow$ $x^2 - 2 = 0$?

$\mathbb{P}, \mathbb{Q}^c = \{$ numbers not in the form of $p/q \}$ $\longrightarrow$ π, e ?

Transcendental numbers

Terminating $= \dfrac{p}{2^m 5^n}$ format $m, n \in \mathbb{Z}^+$

Real numbers $\rightarrow \mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ $\longrightarrow$ $x^2 + 2 = 0$

Complex number $\longrightarrow$ $\mathbb{C} : \{ a + bi,\ a, b \in \mathbb{R} \}$

Properties

(1) Closure $\rightarrow$ Integer closed under addition

(2) Commutative $\rightarrow$ $x + y = y + x$ / $x \cdot y = y \cdot x$

(3) Associative $\longrightarrow$ $x + (y + z) = (x + y) + z$ / $x(yz) = (xy)z$

(4) Identity $<$ additive, multiplicative $\}$ zero, one

(5) Distributive $\longrightarrow$ $x(y + z) = xy + xz$ | $(x + y)z = xz + yz$

Properties of divisibility

(1) $x | y$ and $y | z \Rightarrow x | z$

(2) $x | y$ and $x | z \Rightarrow x | (my + nz)$, $m, n \in \mathbb{Z}$

(3) $x | y$ and $y | x \Rightarrow x = \pm y$

(4) $x | y$, $x, y > 0$, then $x \leq y$

(5) $x | y \Rightarrow x | yz$, for any $z \in \mathbb{Z}$

(6) $x | y$ iff $nx | ny$ for any $n \in \mathbb{Z}, n \neq 0$

Numbers

Division algorithm $\longrightarrow$ $a = bq + r$, $0 \leq r < b$

HCF / GCD $\longrightarrow$ $c|a$ & $c|b$, $c \rightarrow$ common divisor

GCD (1) $d|a$ & $d|b$

(2) If $c|a$ & $c|b$, then $c \mid d$

12, 18
$c = \{2, 3, 6\}$
$d = 6$
$c_1|d$, $c_2|d$

Properties

(1) If $(b, c) = g$, and d is any common divisor, then $d|g$

(2) For $m > 0$, $(mb, mc) = m(b, c)$, $m \in \mathbb{Z}$

(3) If $d|b, d|c, d > 0$ then $\left(\frac{b}{d}, \frac{c}{d}\right) = \frac{1}{d}(b, c)$

(4) If $(b, c) = g$, then $\left(\frac{b}{g}, \frac{c}{g}\right) = 1$

(5) If $(b, c) = g$, then $\exists\, m, n \in \mathbb{Z}$, $g = mb + nc$

$(12, 18) = 6$, $6 = 12m + 18n$ $m = 5, n = -3$

(6) If $(a, b) = 1$ and $(a, c) = 1$, then $(a, bc) = 1$

(7) If $(a, bc) = 1$, and $(a, b) = 1$, then $(a, c) = 1$

Prime
(1) $P > 1$
(2) P has no divisors except 1 and itself

(+) Integers $= \{1\} \cup \{\text{Composites}\} \cup \{\text{Primes}\}$

$(4, 9) = 1$ neither 4 nor 9 is prime but $(4, 9)$ is co-prime

$(3, 5)$, $(11, 13)$, $(17, 19)$, $(41, 43)$, ... (Twin prime conjecture)

Fundamental theorem of arithmatic

$35 = 1 \times 5 \times 7$
$12 = 1 \times 2^2 \times 3$

$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, number of divisors is given by $d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$

$96 \longrightarrow 2^5 \times 3^1$. $d(96) = (5+1)(1+1) = 12$

$200 \longrightarrow 2 \times (2 \times 5)^2 = 2^3 \times 5^2$ $d(200) = (3+1)(2+1) = 12$

$120 \longrightarrow 2^3 \times 3 \times 5$ $d(120) = 4 \times 2 \times 2 = 16$

Numbers

Properties of $d(n)$

(1) for square numbers, $d(n) =$ odd

$9 = 3^2 \qquad d(9) = 2+1 = 3$

(2) if n is not a perfect square, $d(n) =$ even

$$\sigma(n) = \left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_k^{\alpha_k+1}-1}{p_k-1}\right) = \text{sum of factors}$$

GIF Greatest Integer function (step function) $[x]$

$[8.72] = 8$, $[7.02] = 7$,

$[-2.5] = -3 \qquad [-6.87] = -7$

Fractional part function $\{x\} \qquad \{8.72\} = 0.72$

$x = [x] + \{x\} \qquad -6.87 = -7 + \{x\}$

$\Rightarrow \{x\} = \underline{0.13}$

GIF

(1) $[x+m] = [x] + m$, if m is an integer

(2) $[x] + [y] \leq [x+y] \leq [x] + [y] + 1$

$[5.6] \quad [6.5] \qquad [12.1]$

$\downarrow \quad \downarrow \qquad \downarrow$

$5 + 6 \neq 12$

(3) $[x] + [-x] = 0$ if $x \in \mathbb{Z}$

else, $[x] + [-x] = -1$

(4) $\left[\frac{[x]}{m}\right] = \left[\frac{x}{m}\right]$ if $x \in \mathbb{Z}$

→ If $a \in \mathbb{R}$ & $c \in \mathbb{N}$, then $\left[\frac{[a]}{c}\right] = \left[\frac{a}{c}\right]$

→ If $x \in \mathbb{R}$, $\left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right] = [x]$

Q Find the highest power of 3 contained in 1000!

$p = 3$, $n = 1000$

$\left[\frac{n}{p}\right] = \left[\frac{1000}{3}\right] = \underline{333}$

$\left[\frac{n}{p^2}\right] = \left[\frac{333}{3}\right] = \underline{111}$

$\left[n/p^3\right] = \left[111/3\right] = \underline{37}$

$\left[n/p^4\right] = \underline{12}$

$\left[n/p^5\right] = \underline{4}$

$\left[n/p^6\right] = \underline{1}$

$\left[\frac{n}{p^7}\right] = 0$

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Numbers

Let a, b be odd integers. If 4 doesn't divide $(a-b)$ show 4 doesn't divide (a^3-b^3)

Suppose $4 \nmid (a-b)$, then

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$= (a-b)\left((a+b)^2-3ab\right)$$

If a, b are odd, $(a+b)^2$ is even & $3ab$ is odd

$\therefore \quad (a+b)^2 - 3ab = \text{odd}$

$a^3-b^3 = $ (not a multiple of 4) $\times$ (odd number)

$\Rightarrow \quad (a^3-b^3)$ is not divisible by 4

Show that the sum of first n natural numbers $(n \geq 3)$ is never prime.

Sum of naturals upto $n = \dfrac{n(n+1)}{2}$

This is clearly composite — If n is even, $n(n+1)$ is even

If n is odd, $(n+1)$ is even, $n(n+1)$ is even

Numbers