02 September 2023 08:20

Application of Derivatives / Integrals

The sum of the absolute minimum and the absolute maximum values of the function $f(x) = |3x - x^2 + 2| - x$ in the interval $[-1, 2]$ is:

(A) ~~(A)~~ $\frac{\sqrt{17}+3}{2}$ (B) $\frac{\sqrt{17}+5}{2}$

(C) 5 (D) $\frac{9-\sqrt{17}}{2}$

$|6-4+2| - 2 = 2.$

$|g(x)|_{min} = 0$

$f(x) = |g(x)| - x.$

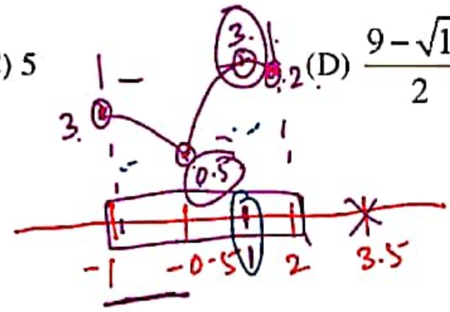
$g(x) = 0$

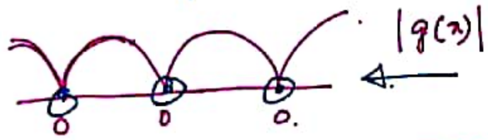
$|-3-1+2|$

$2 \quad -(-1)$

$0 - (x)$

$\frac{3-\sqrt{17}}{2}$





$\sqrt{17} \approx 4$

$-x^2+3x+2=0.$ $x = \frac{-3 \pm \sqrt{3^2+8}}{-2}$

$\cancel{\frac{3+4}{2}}, \frac{3-4}{2}$

$\cancel{3.5}, -0.5$

$\boxed{x = \frac{3 \pm \sqrt{17}}{2}}$

$= \frac{-3 \pm \sqrt{17}}{-2}.$

$f(x) = -(-x^2+3x+2) - x.$ $x < \frac{3-\sqrt{17}}{2}$

$= -x^2+3x+2 - x$ $x \geq \frac{3-\sqrt{17}}{2}$ for $x < 0$

$f'(x) < 0$

$x=1$

$f(x) = -1+2+2$

$= 3$

$f(x) = \boxed{x^2-4x-2}$

$= -x^2+2x+2.$

$f'(x) = 2x-4 = 0 \Rightarrow x = 2 \not{>} 0$

$= -2x+2 = 0 \Rightarrow x=1$

$f(x)_{max} = 3$ $f(x)_{min} = \frac{\sqrt{17}-3}{2}.$

$\text{Sum} = \frac{\sqrt{17}-3}{2} + 3 = \frac{\sqrt{17}+3}{2}.$

Let S be the set of all the natural numbers, for which the line $\frac{x}{a}+\frac{y}{b}=2$ is a tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at the point (a, b), $ab \neq 0$. Then:

(A) $S = \phi$ (B) $n(S) = 1$

(C) $S = \{2k : k \in N\}$ (D) $S = N$

$y = mx + c$

What values can n take? $\Rightarrow$ Set S

$\text{Slope} = \left(-\frac{b}{a}\right) = \left(\frac{dy}{dx}\right)_{\text{curve}}$

$\left(\frac{1}{a^n}\right) n x^{n-1} + \left(\frac{1}{b^n}\right) n y^{n-1} \frac{dy}{dx} = 0.$

$x=a, y=b$ $\quad \frac{1}{a^n} \cdot n a^{n-1} + \frac{1}{b^n} \cdot n b^{n-1} \frac{dy}{dx} = 0.$

for $x=a, y=b$

$$\frac{1}{a^n}\cdot n a^{n-1} + \frac{1}{b^n}\cdot n b^{n-1}\frac{dy}{dx} = 0.$$

$$\frac{1}{a} + \frac{1}{b}\frac{dy}{dx} = 0. \quad \boxed{\frac{dy}{dx} = \left(\frac{-b}{a}\right)}$$ is independent of n.

Let $f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10$, $x \in [-1, 1]$. If $[a, b]$ is the range of the function then $(4a - b)$ is equal to:

(A) 11 (B) $11 - \pi$ (C) $11 + \pi$ (D) $15 - \pi$

$x \in [-1, 1]$

$R \in [a, b]$

$4a - b = ?$

$x=-1$ $x=1$

$\frac{dy}{dx} = ?$

$y = \cos^{-1}x.$

$x = \cos y.$

$1 = -\sin y \frac{dy}{dx}.$

$\frac{dy}{dx} = \frac{-1}{\sin y}$

$= \frac{-1}{\sqrt{1-\cos^2 y}}$

$y = \cot^{-1}x.$

$\cot y = x.$

$-\operatorname{cosec}^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y}.$

$f'(x) > 0$ or $f'(x) < 0. \Rightarrow f(x)\uparrow$ or $\boxed{f(x)\downarrow}$.

$$f'(x) = 2\left(\frac{-1}{\sqrt{1-x^2}}\right) + 4\left(\frac{-1}{1+x^2}\right) - 6x - 2.$$

$$= -\left[\frac{2}{\sqrt{1-x^2}} + \frac{4}{1+x^2} + 6x + 2\right]$$

$f'(x) < 0$

$$f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10.$$

$$f(-1) = 2\cos^{-1}(-1) + 4\cot^{-1}(-1) - 3(-1)^2 - 2(-1) + 10. = 2\pi + 4\cdot\frac{3\pi}{4} - 3 + 2 + 10.$$

$$= 5\pi + 9 = b$$

$$f(1) = 2\cos^{-1}(1) + 4\cot^{-1}(1) - 3(1)^2 - 2(1) + 10 = 0 + 4\cdot\frac{\pi}{4} - 3 - 2 + 10.$$

$$= \pi + 5 = a.$$

$$4a - b = 4\pi + 20 - 5\pi - 9 = -\pi + 11$$

Consider a cuboid of sides 2x, 4x and 5x and a closed hemisphere of radius r. If the sum of their surface areas is a constant k, then the ratio x : r, for which the sum of their volumes is maximum, is :

(A) 2 : 5 (B) 19:45 (C) 3 : 8 (D) 19 : 15

SA of cuboid $= 2(LB+BH+LH)$

$= 2(8x^2+20x^2+10x^2)$

$= 76x^2$

SA of closed hemisphere $= 3\pi r^2$

Vol of cuboid $= 40x^3$.

" " hemisphere $= \frac{2}{3}\pi r^3$

$$\boxed{76x^2+3\pi r^2 = K.}$$

$$V = 40x^3 + \frac{2}{3}\pi r^3.$$

$$r^2 = \frac{K-76x^2}{3\pi}.$$

$$\boxed{r = \left(\frac{K-76x^2}{3\pi}\right)^{1/2}}$$

$$V = 40x^3 + \frac{2}{3}\pi \cdot \frac{1}{(3\pi)^{3/2}}\left(K-76x^2\right)^{3/2}.$$

$$\frac{dV}{dx} = 0.$$

$$120x^2 + \frac{\cancel{2}}{9\sqrt{3\pi}} \cdot \frac{3}{\cancel{2}}\left(K-76x^2\right)^{1/2} \cdot (-152x) = 0$$

$$120x^{\cancel{2}} = 152x\boxed{\left(\frac{K-76x^2}{3\pi}\right)^{1/2}} = 152\cancel{x}r.$$

$$\frac{x}{r} = \frac{152}{120} = \frac{38}{30} = \frac{19}{15}$$

If y = y(x) is the solution of the differential equation $x\frac{dy}{dx}+2y = xe^x, y(1)=0$, then the local maximum value of the function $z(x) = x^2y(x)-e^x$, $x \in R$ is :

(A) $1-e$ (B) 0 (C) $\frac{1}{2}$ (D) $\frac{4}{e}-e$

$Z_{max} = ?$

$$x\frac{dy}{dx} + 2y = xe^x.$$

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = e^x.$$

$$\frac{dy}{dx} + Py = Q$$

$\int \frac{2}{x}dx$ $2\log x$.

$y(IF) = \int Q(IF)dx$.

$\frac{dy}{dx} + Py = Q$

$y \cdot (IF) = \int Q \cdot (IF)\, dx.$

$IF = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\log x}$

$= e^{\log(x^2)} = x^2$

$yx^2 = \int e^x x^2 dx.$ $\quad \int u\,dv = uv - \int v\,du.$

$yx^2 = x^2 e^x - \int 2x e^x dx.$ $\quad v = \int e^x dx = e^x.$

$= x^2e^x - 2\left[\int x e^x dx\right] = x^2 e^x - 2\left[xe^x - \int e^x dx\right] = x^2e^x - 2(xe^x - e^x) + C$

$yx^2 = e^x(x^2 - 2x + 2) + C.$

$y(1) = 0.$ $\quad 0 = e(1-2+2) + C \Rightarrow C = -e.$

$\boxed{yx^2 = e^x(x^2-2x+2) - e.}$

$z = x^2y - e^x = e^x(x^2-2x+2) - e - e^x.$

$z = e^x(x^2-2x+1) - e. = e^x(x-1)^2 - e.$

$\frac{dz}{dx} = e^x(x-1)^2 + e^x \cdot 2(x-1) = 0$

$e^x(x-1)(x-1+2) = 0.$

$e^x(x+1)(x-1) = 0$

$\boxed{x = -1, 1}$

for $x=1$

$z = -e$

for $x=-1$

$z = e^{-1} 4 - e$

max $\boxed{z = \frac{4}{e} - e.}$

The number of distinct real roots of $\underbrace{x^4 - 4x + 1}_{f(x)} = 0$ is :

(A) 4 (B) 2

(C) 1 (D) 0

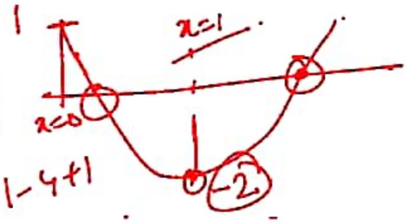
No of sign changes of $f(x)$ $= 2.$

$\therefore$ No of +ve roots $= 2, 0$

$f(-x) = x^4 + 4x + 1$

No of sign changes of $f(-x) = 0$

$\therefore$ No of -ve roots $= 0$



for $x=1$

$f(x) = 1-4+1$

$f(x) = 1-4+1$

$= -2$ $\quad f'(x) = 4x^3 - 4 = 4(x-1)(x^2+x+1)$

$\therefore$ No of $-$ve roots $= 0$

for $x=0$ $f(x)=1$. $\quad f'(x) = 0 \Rightarrow x = 1$ min $\quad f''(x) = 12x^2 > 0$