Rules for Particular Integrals:-

Q. Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x$ ... [Find PI]

Notation: $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$

$\therefore \quad D^2y - 5Dy + 6y = e^x$

$\Rightarrow \underbrace{(D^2 - 5D + 6)}\, y = e^x$

$\Rightarrow f(D)\, y = e^x$ [$f(D)$: polynomial in D]

In general: $f(D)\, y = Q$ [$f(D)$: polynomial in D, Q is fn of x]

representation of my differential eqn.

$\therefore$ For P.I $\boxed{y = \frac{1}{f(D)} \cdot Q}$

Note: $D = \frac{d}{dx}$, $\frac{1}{D} = \int$ ... [Representation & Interpretation]

Case I: If RHS $= e^{\alpha x}$

Then: P.I : $y = \begin{cases} \frac{1}{f(D)} \cdot e^{\alpha x} ; & \text{Put } D = \alpha, \text{ if } f(\alpha) \neq 0. \\ x \cdot \frac{1}{\frac{d}{dD}[f(D)]} \cdot e^{\alpha x} ; & \text{if } f(\alpha) = 0. \end{cases}$

Q. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x$, Here $\alpha = 1$

$\Rightarrow (D^2 - 5D + 6)\, y = e^x$

$f(D)\, y = e^x$ [$f(D) = D^2 - 5D + 6$] [$\alpha = 1$]

$\Rightarrow (D^2 - 5D + 6)\, y = e^{x}$.

$f(D)\cdot y = e^{x}$ $\quad [f(D) = D^2 - 5D + 6]$ $\quad [\alpha = 1]$

For P.I : $y = \dfrac{1}{f(D)} e^{x} = \dfrac{1}{D^2 - 5D + 6} e^{x} = \dfrac{1}{1-5+6} e^{x} = \dfrac{e^{x}}{2}$.

Case II: $\dfrac{1}{f(D^2)} \sin \alpha x$ or $\dfrac{1}{f(D^2)} \cos \alpha x$.

For P.I : $\left\{\begin{array}{l} \text{Put } D^2 = -\alpha^2, \text{ provided } f(-\alpha^2) \neq 0 . \\ y = x \cdot \dfrac{1}{\frac{d}{dD}[f(D^2)]} \sin \alpha x , \text{ if } f(-\alpha^2) = 0 . \end{array}\right.$

$[\alpha = 1]$

Q. Solve : $(D^4 + 2D^3 - 3D^2)\, y = 4 \sin x$. [Find P.I]

$f(D) = D^4 + 2D^3 - 3D^2$.

$\therefore$ For P.I : $y = \dfrac{1}{f(D)} \cdot 4 \sin x = \dfrac{1}{D^4 + 2D^3 - 3D^2} \cdot 4 \sin x$.

$= \dfrac{1}{D^2 [D^2 + 2D - 3]} \cdot 4 \sin x$

Put $D^2 = -1$

$= \dfrac{1}{(-1)[-1 + 2D - 3]} \cdot 4 \sin x$

$= \dfrac{1}{(-1)[2D - 4]} \cdot 4 \sin x$

$= \dfrac{1}{\cancel{2}(-1)[D - 2]} \cdot \cancel{4}^{\,2} \sin x$

$= -2 \cdot \dfrac{1}{(D-2)} \cdot \sin x$.

$= -2 \cdot \dfrac{(D+2)}{(D^2 - 4)} \cdot \sin x$

$$= -2 \cdot \frac{(D+2)}{(D^2-4)} \cdot \sin x$$

Put $D^2 = -1$

$$= \frac{-2(D+2)}{-1-4} \sin x$$

$$= \frac{2}{5}[D+2] \sin x$$

$$= \frac{2}{5}\left[\frac{d}{dx}\sin x + 2\sin x\right]$$

$$= \frac{2}{5}[\cos x + 2\sin x]$$

Case III : If $RHS = e^{\alpha x} \cdot g(x)$.

P.I : $y = \frac{1}{f(D)} \cdot e^{\alpha x} g(x) = e^{\alpha x} \cdot \frac{1}{f(D+\alpha)} g(x)$.

Q. Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}$. $\left[\begin{array}{l} \alpha = 3 \\ g(x) = x^2 \end{array}\right]$

$\Rightarrow D^2 y - 2Dy + y = x^2 e^{3x}$

$\Rightarrow (D^2 - 2D + 1) y = x^2 e^{3x}$.

$\Rightarrow f(D) \cdot y = x^2 e^{3x}$ [where $f(D) = D^2 - 2D + 1$] .

For P.I : $y = \frac{1}{f(D)} \cdot x^2 e^{3x}$.

$$= e^{3x} \frac{1}{f(D+3)} \cdot x^2$$

$f(D) = D^2 - 2D + 1$.

$f(D+3) = (D+3)^2 - 2(D+3) + 1$.

$= D^2 + 9 + 6D - 2D - 6 + 1$

$= D^2 + 4 + 4D = (D+2)^2$

P.I $y = e^{3x} \cdot \frac{1}{(D+2)^2} \cdot x^2$

P.I $\quad y = e^{3x} \cdot \frac{1}{(D+2)^2} \cdot x^2$

$$= \frac{e^{3x}}{4} \cdot \frac{1}{\left(1+\frac{D}{2}\right)^2} x^2$$

[Express $f(D) = (1 + D + \cdots)^n$]

$$= \frac{e^{3x}}{4}\left[1+\frac{D}{2}\right]^{-2} x^2$$

[$\left(1+\frac{D}{2}\right)^{-2}$ .. use Binomial Th]

$$= \frac{e^{3x}}{4}\left[1 - 2\left(\frac{D}{2}\right) + 3\left(\frac{D^2}{4}\right) + 4 \cdot \frac{D^4}{16}\right] x^2$$

$$= \frac{e^{3x}}{4}\left[1 - D + \frac{3}{4}D^2 + \cdots\right] x^2$$

$D \downarrow \frac{d}{dx} \qquad D^2 \downarrow \frac{d^2}{dx^2}$

$$= \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{4}\cdot 2\right] = \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$$