Real Analysis

Consider a set of numbers:

Eg: $S = \{1, 4, 9, 16, \ldots\}$ ---- $\{n^2\}$ -- [all terms ≥ 1] — lower bound. 0.9, 0.8

(*) $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ ---- $\{\frac{1}{n}\}$ -- [all terms ≤ 1] — upper bound.

$S = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$

↳ 1 = lower bounds, 2 = upper bounds.

Lower & Upper bound:

Consider $x \in S$. A no. l can be said to be lower bound of S if $x \geq l \ \forall x \in S$.

Consider $x \in S$. A no. 'u' can be said to be upper bound of S if $x \leq u \ \forall x \in S$.

Note: Upper and Lower bounds need not be unique.

If a 'S' has an upper bound $\Rightarrow$ S is bounded above.

If a 'S' has a lower bound $\Rightarrow$ S is bounded below.

If set 'S' has both upper & lower bounds $\Rightarrow$ S is bounded set.

" Least Upper Bound of the Set" → Supremum of set S $\Rightarrow \sup(S)$

" Greatest Lower Bound of the Set" → Infimum of set S $\Rightarrow \inf(S)$

(i) If a set is bounded above, it has a supremum. (Supremum of a set is unique)

(ii) If a set is bounded below, it has an infimum. (Infimum of a set is unique)

Q. Let $a_n = \begin{cases} 2 + (-1)^{\frac{n-1}{2}}, & n \text{ is odd} \\ 1 + \frac{1}{2n}, & n \text{ is even} \end{cases}, \quad n \in \mathbb{N}.$

$(-1)^{\frac{n-1}{2}} = \pm 1 = 3$ → 3, 1, 3, 1, …… (max, min)

$$\left\{ \quad 1 + \frac{1}{2^n}, \quad n \text{ is even} \right. , \quad n \in \mathbb{N}.$$

$\hookrightarrow \ldots \left(1+\frac{1}{4}\right), \left(1+\frac{1}{16}\right) \ldots$

$\hookrightarrow$ min $= 1$.

Find $\sup(a_n)$, $\inf(a_n)$.

$\sup(a_n) = 3$. ✓

$\inf(a_n) = 1$ ✓.

Note: Consider 2 sets S, T s.t $S \subseteq T$ $(S, T \subseteq \mathbb{R})$ Then:

(i) If T is bounded above, $\sup(S) \leqslant \sup(T)$

(ii) If T is bounded below, $\inf(T) \leqslant \sup(S)$.

(i) Let $\sup(T) = k$. Then, $\forall y \in T$, $y \leq k$.

Since $S \subseteq T$, $x \in S \Rightarrow x \in T$, so $x \leq k$

$\Rightarrow \sup(S) \leq k = \sup(T) \Rightarrow \sup(S) \leq \sup(T)$.